On transversal slices for modules over representation finite algebras

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Notations and general assumptions

- $\Bbbk$ is an algebraically closed field of arbitrary characteristic.
- Variety is a (not necessarily reduced) scheme of finite type over $\Bbbk$. All considered points are closed.
- Algebras are finitely generated associative $\Bbbk$-algebras with a unit.
- Given an algebra Λ, $\mathrm{Mod}(\Lambda)$ denotes the category of left Λ-modules and $\mathrm{mod}(\Lambda)$ is its full subcategory of finite dimensional modules.
- Quivers $Q = (Q_0, Q_1, s, t)$ are assumed to be finite.
- A bound quiver is a pair (Q, I), where Q is a quiver and I is a (not necessarily admissible) two-sided ideal in the path algebra $\Bbbk Q$.
- $\mathrm{Rep}(Q)$ is the category of representations of Q, $\mathrm{rep}(Q)$ is its full subcategory of finite dimensional representations. We define $\mathrm{Rep}(Q, I)$ and $\mathrm{rep}(Q, I)$ in a similar way.

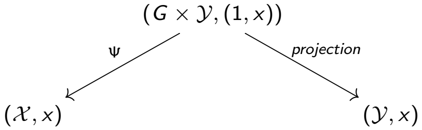
Introduction to transversal slices

Definition

Let G be an algebraic group acting regularly on a (possibly not reduced) variety $\mathcal{X}$, and $x \in \mathcal{X}$. A **transversal slice** in $\mathcal{X}$ to the orbit $G \cdot x$ at the point x is a subvariety $\mathcal{Y}$ of $\mathcal{X}$ satisfying:

- $x \in \mathcal{Y}$;
- the morphism $\Psi\colon G \times \mathcal{Y} \to \mathcal{X}$, $(g, y) \mapsto g \cdot y$, is smooth;
- $\dim \mathcal{Y}$ is minimal with respect to the above.

- If $\mathcal{Y}$ is a transversal slice in $\mathcal{X}$ at x, then the pointed varieties $(\mathcal{X}, x)$ and $(\mathcal{Y}, x)$ are **smoothly equivalent**, i.e. they are connected via smooth morphisms:



  Thus $(\mathcal{X}, x)$ and $(\mathcal{Y}, x)$ share many local geometric properties of varieties (like smoothness, normality, being Cohen-Macaulay, Gorenstein, ...)
- Let $\mathcal{X}$ and $\mathcal{X}'$ be G-varieties, $F\colon \mathcal{X} \to \mathcal{X}'$ be a G-equivariant morphism, and $x \in \mathcal{X}$. If $\mathcal{Y}'$ is a transversal slice in $\mathcal{X}'$ at $F(x)$, then $F^{-1}(\mathcal{Y}')$ is a transversal slice in $\mathcal{X}$ at x.

- Let $\mathcal{Y}$ be a transversal slice in $\mathcal{X}$ at x, and consider the following maps:

$$\mu\colon G \to \mathcal{X},\ \mu(g) = g \cdot x, \qquad \text{and} \qquad \mathcal{T}_{1,\mu}\colon \mathcal{T}_{1,G} \to \mathcal{T}_{x,\mathcal{X}}.$$

  Then

$$\mathcal{T}_{x,\mathcal{X}} = \mathcal{T}_{x,\mathcal{Y}} \oplus \operatorname{Im}(\mathcal{T}_{1,\mu}), \tag{1}$$

  and

$$\dim_x \mathcal{Y} = \dim_x \mathcal{X} - \dim_{\Bbbk} \operatorname{Im}(\mathcal{T}_{1,\mu}).$$

Question

How to construct transversal slices?

- Assume first that x is a smooth point of a G-variety $\mathcal{X}$. We choose a locally closed smooth subvariety $x \in \mathcal{Y} \subseteq \mathcal{X}$ satisfying (1). Then $\Psi\colon G \times \mathcal{Y} \to \mathcal{X}$ is smooth at (g, x) for any $g \in G$. Replacing $\mathcal{Y}$ by its open neighbourhood of x if necessary, we get a transversal slice in $\mathcal{X}$ at x.
- If $\mathcal{X}$ is an affine G-variety, then there is a smooth affine G-variety $\mathcal{X}'$ together with a G-equivariant closed immersion

$$F\colon \mathcal{X} \to \mathcal{X}'.$$

  Given a point $x \in \mathcal{X}$, we choose a transversal slice $\mathcal{Y}'$ in $\mathcal{X}'$ at $F(x)$. Then $F^{-1}(\mathcal{Y}')$ is a transversal slice in $\mathcal{X}$ at x.

Varieties of representations of quivers

- Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $\mathbf{d} = (d_u) \in \mathbb{N}^{Q_0}$ a dimension vector. We define

$$\operatorname{rep}_Q(\mathbf{d}) = \{V = (V_u, V_\alpha) \in \operatorname{rep}(Q) | \; V_u = \Bbbk^{d_u}\} = \prod_{\alpha \in Q_1} \operatorname{Hom}_{\Bbbk}(\Bbbk^{d_{s(\alpha)}}, \Bbbk^{d_{t(\alpha)}})$$

$$= \prod_{\alpha \in Q_1} \mathbb{M}_{d_{t(\alpha)} \times d_{s(\alpha)}}(\Bbbk).$$

- The group $\operatorname{GL}(\mathbf{d}) = \prod_{u \in Q_0} \operatorname{GL}(d_u)$ acts on $\operatorname{rep}_Q(\mathbf{d})$:

$$(g * V)_\alpha = g_{t(\alpha)} \cdot V_\alpha \cdot g_{s(\alpha)}^{-1},$$

  where $g = (g_u)_{u \in Q_0}$ and $V = (V_\alpha)_{\alpha \in Q_1}$.
- If $M \in \operatorname{rep}_Q(\mathbf{d})$, then

$$\operatorname{GL}(\mathbf{d}) * M = \{V \in \operatorname{rep}_Q(\mathbf{d}) | \; V \simeq M\}.$$

- Given $I \lhd \Bbbk Q$, we denote by $\operatorname{rep}_{Q,I}(\mathbf{d})$ (possibly non reduced) closed $\operatorname{GL}(\mathbf{d})$-subvariety of $\operatorname{rep}_Q(\mathbf{d})$ consisting of the representations of Q which are annihilated by I.

Transversal slice for representations of quivers

- We want to construct a transversal slice in $\mathrm{rep}_{Q,I}(\mathbf{d})$ at a point N.
- Since $\mathrm{rep}_{Q,I}(\mathbf{d}) \subseteq \mathrm{rep}_Q(\mathbf{d})$, and the latter is smooth, we start with a smooth subvariety $\mathcal{Y} \subseteq \mathrm{rep}_Q(\mathbf{d})$ such that $N \in \mathcal{Y}$ and

$$\mathcal{T}_{N,\mathrm{rep}_Q(\mathbf{d})} = \mathcal{T}_{N,\mathcal{Y}} \oplus \mathrm{Im}(\mathcal{T}_{1,\mu}), \qquad \text{where} \qquad \mu\colon \mathrm{GL}(\mathbf{d}) \to \mathrm{rep}_Q(\mathbf{d}), \quad g \mapsto g * N.$$

- The tangent space $\mathcal{T}_{N,\mathrm{rep}_Q(\mathbf{d})}$ can be identified with

$$\mathbb{Z}^1_Q(N,N) \simeq \prod_{\alpha \in Q_1} \mathrm{Hom}_{\Bbbk}(N_{s\alpha}, N_{t\alpha}) = \prod_{\alpha \in Q_1} \mathrm{Hom}_{\Bbbk}(\Bbbk^{d_{s\alpha}}, \Bbbk^{d_{t\alpha}}) \simeq \mathrm{rep}_Q(\mathbf{d}).$$

- Moreover,

$$\mathcal{T}_{1,\mu} : \prod_{u \in Q_0} \mathbb{M}_{d_u}(\Bbbk) \to \mathbb{Z}^1_Q(N,N), \quad (h_u)_{u \in Q_0} \mapsto (h_{t(\alpha)} \cdot N_\alpha - N_\alpha \cdot h_{s(\alpha)})_{\alpha \in Q_1}.$$

- Hence,

$$\mathrm{Ker}(\mathcal{T}_{1,\mu}) = \mathrm{End}_Q(N), \qquad \mathrm{Im}(\mathcal{T}_{1,\mu}) = \mathbb{B}^1_Q(N,N).$$

- We choose $\mathcal{Y} := N + \mathcal{C}$, where $\mathcal{C}$ is a $\Bbbk$-linear complement:

$$\mathbb{Z}^1_Q(N,N) = \mathbb{B}^1_Q(N,N) \oplus \mathcal{C} \qquad (\text{thus } \mathcal{C} \simeq \mathrm{Ext}^1_Q(N,N)).$$

- Some open neighbourhood $\mathcal{U}$ of N in $\mathcal{Y}$ is a transversal slice in $\mathrm{rep}_Q(\mathbf{d})$ at N.
- Consequently, $\mathcal{U} \cap \mathrm{rep}_{Q,I}(\mathbf{d})$ is a transversal slice in $\mathrm{rep}_{Q,I}(\mathbf{d})$ at N.

Transversal slices in representation-finite type

- Assume $\Lambda = \Bbbk Q/I$ is a finite dimension algebra having only finitely many indecomposable modules (up to isomorphism): $Y_1, \dots, Y_n \in \mathrm{mod}(\Lambda) = \mathrm{rep}(Q, I)$.
- Aim: to define transversal slices to all orbits of representations of (Q, I), in a uniform way.
- We choose a $\Bbbk$-linear complement

$$\mathbb{Z}^1_Q(Y_i, Y_j) = \mathbb{B}^1_Q(Y_i, Y_j) \oplus \mathcal{C}_{i,j}, \qquad \text{for all } i, j \leq n.$$

- Let $Y = \bigoplus_i Y_i$ and $\mathcal{C} = \bigoplus_{i,j} \mathcal{C}_{i,j} \subseteq \mathbb{Z}^1_Q(Y, Y)$.
- The tensor algebra $T_{\Bbbk^n}(\mathbb{D}\mathcal{C}) \simeq T_{\Bbbk^n}\left(\mathbb{D}\,\mathrm{Ext}^1_Q(Y, Y)\right)$ can be viewed as the path algebra $\Bbbk\widehat{Q}$ of a quiver $\widehat{Q}$ with the set of vertices $\widehat{Q}_0 = \{1, \dots, n\}$. This leads to an exact functor

$$F\colon\ \mathrm{rep}(\widehat{Q}) \to \mathrm{rep}(Q)$$

mapping the standard simple representations S_i to Y_i, $i \leq n$, and a closed immersion

$$F(\mathbf{m})\colon\ \mathrm{rep}_{\widehat{Q}}(\mathbf{m}) \to \mathrm{rep}_Q(\phi(\mathbf{m})), \qquad \text{where } \phi(\mathbf{m}) = \sum m_i \cdot \mathbf{dim}\, Y_i,$$

such that the restriction of $\mathrm{Im}(F(\mathbf{m}))$ to an appropriate open subvariety is a transversal slice in $\mathrm{rep}_Q(\phi(\mathbf{m}))$ at $F(\mathbf{m})(0) = \bigoplus_i (Y_i)^{m_i}$, for any $\mathbf{m} \in \mathbb{N}^{\widehat{Q}_0}$.
- The ideal $I \lhd \Bbbk Q$ corresponds to an ideal $\widehat{I} \lhd \Bbbk\widehat{Q}$, such that F and $F(\mathbf{m})$ restrict to

$$F'\colon\ \mathrm{rep}(\widehat{Q}, \widehat{I}) \to \mathrm{rep}(Q, I) \qquad \text{and} \qquad F'(\mathbf{m})\colon\ \mathrm{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) \to \mathrm{rep}_{Q,I}(\phi(\mathbf{m})),$$

and the restriction of $\mathrm{Im}(F'(\mathbf{m}))$ to an appropriate open subvariety is a transversal slice in $\mathrm{rep}_{Q,I}(\phi(\mathbf{m}))$ at $F'(\mathbf{m})(0) = \bigoplus_i (Y_i)^{m_i}$, for any $\mathbf{m} \in \mathbb{N}^{\widehat{Q}_0}$.

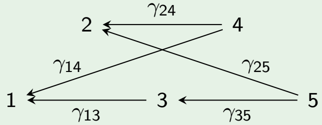
Example

Let $Q:\ 1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3$ and $I = \langle \alpha\beta \rangle$. Then

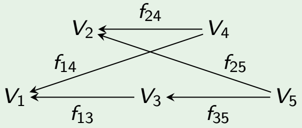
$$Y = Y_1 \oplus \cdots \oplus Y_5 = \qquad \Bbbk^2 \xleftarrow{\left[\begin{smallmatrix} 0&0&0\\ 1&0&0 \end{smallmatrix}\right]} \Bbbk^3 \xleftarrow{\left[\begin{smallmatrix} 0&0\\ 0&0\\ 1&0 \end{smallmatrix}\right]} \Bbbk^2.$$

$$\mathbb{B}^1_Q(Y,Y) = \left\{ \Bbbk^2 \xleftarrow{\left[\begin{smallmatrix} *&0&0\\ *&*&t \end{smallmatrix}\right]} \Bbbk^3 \xleftarrow{\left[\begin{smallmatrix} -t&0\\ *&0\\ *&* \end{smallmatrix}\right]} \Bbbk^2 \right\}, \qquad \mathcal{C} = \left\{ \Bbbk^2 \xleftarrow{\left[\begin{smallmatrix} 0&*&*\\ 0&0&* \end{smallmatrix}\right]} \Bbbk^3 \xleftarrow{\left[\begin{smallmatrix} 0&*\\ 0&*\\ 0&0 \end{smallmatrix}\right]} \Bbbk^2 \right\}.$$

$\widehat{Q}:$



$F\Big($



$$\Big) = V_1 \oplus V_2 \xleftarrow{\left(\begin{smallmatrix} 0 & f_{13} & f_{14}\\ 1 & 0 & f_{24} \end{smallmatrix}\right)} V_2 \oplus V_3 \oplus V_4 \xleftarrow{\left(\begin{smallmatrix} 0 & f_{25}\\ 0 & f_{35}\\ 1 & 0 \end{smallmatrix}\right)} V_4 \oplus V_5.$$

$$\widehat{I} = \langle\, \gamma_{14},\ \gamma_{13}\gamma_{35},\ \gamma_{24},\ \gamma_{25} \,\rangle.$$

Corollary

Let $\Lambda = \Bbbk Q/I$ be a representation-finite algebra. Then there is a bound quiver $(\widehat{Q}, \widehat{I})$ together with an exact functor $F' \colon \operatorname{rep}(\widehat{Q}, \widehat{I}) \to \operatorname{rep}(Q, I)$ and closed immersions

$$F'(\mathbf{m}) \colon \operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) \to \operatorname{rep}_{Q,I}(\phi(\mathbf{m}))$$

such that for any $N \in \operatorname{rep}(Q, I)$ there is $\mathbf{m}$ such that

- $N' := F'(\mathbf{m})(0) \simeq N$,
- $\dim_0 \operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) = \dim_{N'} \operatorname{rep}_{Q,I}(\phi(\mathbf{m})) - \dim \operatorname{GL}(\phi(\mathbf{m})) * N'$,
- $\operatorname{GL}(\phi(\mathbf{m})) \times \operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) \to \operatorname{rep}_{Q,I}(\phi(\mathbf{m}))$, $(g, L) \mapsto g * F'(\mathbf{m})(L)$, is smooth at $(1, 0)$.

Unfortunately, we miss a representation-theoretic interpretation of $\operatorname{rep}(\widehat{Q}, \widehat{I})$. For instance, we do not know when the images of two points in $\operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m})$ belongs to the same orbit in $\operatorname{rep}_{Q,I}(\phi(\mathbf{m}))$.

Idea is to find a new pair $(\widehat{Q}, \widehat{I})$ which is more closely related to the category $\operatorname{rep}(Q, I)$, and satisfies the above corollary except "closed immersions".

A criterion for smoothness

Let $(\widehat{Q},\widehat{I})$ and (Q,I) be two bounded quivers, and $\mathcal{S}$ be the path category of $(\widehat{Q},\widehat{I})^{op}$. Let

$$\Psi\colon \mathcal{S} \to \operatorname{Rep}(Q,I)$$

be an additive functor such that the composition $\mathcal{S} \xrightarrow{\Psi} \operatorname{Rep}(Q,I) \to \operatorname{Mod}(\Bbbk)$ is isomorphic to a finite direct sum of representable functors $\operatorname{Hom}_{\mathcal{S}}(v,-)$, $v \in \widehat{Q}_0$. Then Ψ induces an exact functor

$$\Phi\colon \operatorname{rep}(\widehat{Q},\widehat{I}) \to \operatorname{rep}(Q,I)$$

together with morphisms

$$\Phi(\mathbf{m})\colon \operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) \to \operatorname{rep}_{Q,I}(\phi(\mathbf{m})), \qquad \phi(\mathbf{m}) = \sum_{v\in\widehat{Q}_0} m_v \cdot \mathbf{dim}\,\Phi(S_v), \quad \mathbf{m} \in \mathbb{N}^{\widehat{Q}_0}.$$

Theorem

The morphism

$$\operatorname{GL}(\phi(\mathbf{m})) \times \operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) \to \operatorname{rep}_{Q,I}(\phi(\mathbf{m})), \qquad (g,L) \mapsto g * \Phi(\mathbf{m})(L),$$

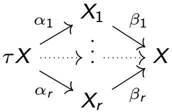
is smooth at $(1,M)$ *provided the map*

$$\operatorname{Ext}^n_{\widehat{Q},\widehat{I}}(M,M) \to \operatorname{Ext}^n_{Q,I}(\Phi(M),\Phi(M)),$$

induced by Φ, *is surjective for* $n=1$ *and injective for* $n=2$.

Representation finite standard algebras

- Let $\Lambda = \Bbbk Q/I$ be a representation finite algebra.
- Let $\mathrm{ind}(\Lambda)$ be a full subcategory of $\mathrm{mod}(\Lambda)$ whose objects form a set of representatives of the isomorphism classes of indecomposable Λ-modules.
- Let (Γ_Λ, τ) denote the Auslander-Reiten quiver of Λ. In particular, $(\Gamma_\Lambda)_0 = \mathit{Objects}(\mathrm{ind}(\Lambda))$.
- The mesh category $\Bbbk(\Gamma_\Lambda)$ of Γ_Λ is a quotient of the path category $\Bbbk[\Gamma_\Lambda]$ modulo mesh relations $\sum \beta_i \alpha_i = 0$:



- We assume that the algebra Λ is **standard**, i.e. there is an equivalence

$$F\colon \Bbbk(\Gamma_\Lambda) \to \mathrm{ind}(\Lambda).$$

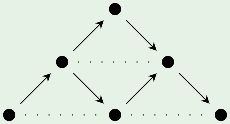
- Given two vertices u, v, and a linear combination ω of paths in Γ_Λ starting at u and terminating at v, we denote by $\overline{\omega}$ its image under the composition

$$\Bbbk[\Gamma_\Lambda] \to \Bbbk(\Gamma_\Lambda) \xrightarrow{F} \mathrm{ind}(\Lambda).$$

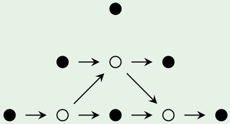
We construct a new translation quiver $(\widehat{\Gamma}_\Lambda, \widehat{\tau})$:

Example (equioriented Dynkin $\mathbb{A}_3$)

Γ_Λ :



$\widehat{\Gamma}_\Lambda$:



The vertices of $\widehat{\Gamma}_\Lambda$:

- **frozen** (bullet) $\{X \mid X \text{ is a vertex of } \Gamma_\Lambda\}$,
- **non-frozen** (circle) $\{X' \mid X \text{ is a non-projective vertex of } \Gamma_\Lambda\}$.

The arrows of $\widehat{\Gamma}_\Lambda$:

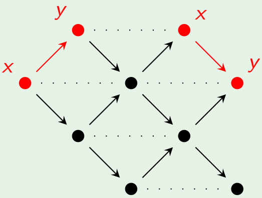
- $\{X' \xrightarrow{\alpha'} Y' \mid X \xrightarrow{\alpha} Y \text{ is an arrow in } \Gamma_\Lambda \text{ and } X, Y \text{ are not projective }\}$,
- $\{\tau X \to X',\ X' \to X \mid X \text{ is a non-projective vertex of } \Gamma_\Lambda\}$.

The translation $\widehat{\tau}$ of $\widehat{\Gamma}_\Lambda$:

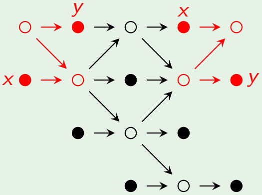
- If $\tau^2 X$ exists in Γ_Λ then $\widehat{\tau}(X') = (\tau X)'$.

Example (Riedtmann's example with 7 indecomposables)

Γ_Λ :



$\widehat{\Gamma}_\Lambda$:



Definition

Let Λ be a standard representation-finite algebra.

- The **regular Nakajima category** $\mathcal{R}_\Lambda$ of the algebra Λ is the mesh category $\Bbbk(\widehat{\Gamma}_\Lambda)$.
- The **singular Nakajima category** $\mathcal{S}_\Lambda$ of the algebra Λ is the full subcategory of $\mathcal{R}_\Lambda$ whose objects are the frozen vertices.

Lemma

$(\mathcal{S}_\Lambda)^{op}$ is isomorphic to the path category of $(\widetilde{Q}, \widetilde{I})$, where $\widetilde{Q}$ is a quiver with the set of vertices $(\widetilde{Q})_0 = (\Gamma_\Lambda)_0$, and the number of arrows in $\widetilde{Q}$ from X to Y equals $\dim_\Bbbk \operatorname{Ext}^1_\Lambda(X, Y)$.

Given a vertex $X \in \Gamma_\Lambda$ and a projective vertex $P \in \Gamma_\Lambda$, we denote by $\Omega(P, X)$ the space of linear combinations of paths $\omega : P \to X$ in Γ_Λ such that among all vertices of ω only the starting one is projective.
We define an additive functor

$$\Psi \colon \Bbbk\left(\widehat{\Gamma}_\Lambda\right) \to \mathrm{Mod}(\Lambda),$$

as follows:

- If X is a frozen vertex, then

$$\Psi(X) = \bigoplus_P \Omega(P, X) \otimes_{\Bbbk} P.$$

  Moreover, we have a canonical surjective Λ-homomorpism

$$\pi_X \colon \Psi(X) \to X, \quad \pi_X(\omega \otimes p) = \overline{\omega}(p).$$

- If X' is non-frozen, then $\Psi(X') = \mathrm{Ker}(\pi_X)$ and $\Psi(X' \to X)$ is a canonical inclusion.
- $\Psi(X' \xrightarrow{\alpha'} Y')$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Psi(X') & \xrightarrow{\Psi(X' \to X)} & \Psi(X) & \xrightarrow{\pi_X} & X & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \Psi(\alpha')} & & \Big\downarrow{\scriptstyle \omega \otimes p \,\mapsto\, \alpha\omega \otimes p} & & \Big\downarrow{\scriptstyle \overline{\alpha}} & & \\
0 & \longrightarrow & \Psi(Y') & \xrightarrow{\Psi(Y' \to Y)} & \Psi(Y) & \xrightarrow{\pi_Y} & Y & \longrightarrow & 0.
\end{array}$$

- $\Psi(\tau X \to X')\colon \bigoplus_P \Omega(P, \tau X) \otimes_{\Bbbk} P \longrightarrow \operatorname{Ker}\left(\bigoplus_P \Omega(P, X) \otimes_{\Bbbk} P \xrightarrow{\pi_X} X\right)$:
  If

$$\begin{array}{ccccc} & & X_1 & & \\ & \overset{\alpha_1}{\nearrow} & & \overset{\beta_1}{\searrow} & \\ \tau X & \cdots\cdots\to & \vdots & \cdots\cdots\to & X \\ & \underset{\alpha_r}{\searrow} & & \underset{\beta_r}{\nearrow} & \\ & & X_r & & \end{array}$$

  is a mesh in Γ_Λ, then

$$\Psi(\tau X \to X')(\omega \otimes p) = -\sum_{X_i \text{ projective}} \beta_i \otimes \overline{\alpha_i \omega}(p) - \sum_{X_i \text{ non-proj.}} \beta_i \alpha_i \omega \otimes p.$$

Proposition

The composition $\mathcal{S}_\Lambda \to \mathcal{R}_\Lambda \xrightarrow{\Psi} \operatorname{Rep}(Q, I) \to \operatorname{Mod}(\Bbbk)$ is isomorphic to a finite direct sum of representable functors $\operatorname{Hom}_{\mathcal{S}_\Lambda}(X, -)$, $X \in (\Gamma_\Lambda)_0$.

Consequently, the composition $\mathcal{S}_\Lambda \to \mathcal{R}_\Lambda \xrightarrow{\Psi} \operatorname{Rep}(Q, I)$ induces an exact functor

$$\Phi\colon \operatorname{rep}(\widetilde{Q}, \widetilde{I}) \to \operatorname{rep}(Q, I)$$

and morphism

$$\Phi(\mathbf{m})\colon \operatorname{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m}) \to \operatorname{rep}_{Q,I}(\phi(\mathbf{m}))$$

for any $\mathbf{m} \in \mathbb{N}^{(\widetilde{Q})_0}$, where $\phi(\mathbf{m}) = \sum_X m_X \cdot \mathbf{dim}_\Lambda X$.

Theorem

Let $\Lambda = \Bbbk Q/I$ *be a standard representation-finite algebra, and* Φ, $\Phi(\mathbf{m})$ *will be as above. Then:*

- $\Phi(S_X) \simeq X$, *where* S_X *is the standard simple representation of* $(\widetilde{Q}, \widetilde{I})$ *at the vertex* $X \in (\widetilde{Q})_0 = (\Gamma_\Lambda)_0$;
- *The map*
$$\mathrm{Ext}^n_{\widetilde{Q},\widetilde{I}}(S_X, S_Y) \to \mathrm{Ext}^n_{Q,I}(X, Y),$$
induced by Φ, *is bijective for any vertices* X, Y, *and* $n \geq 1$.
- *The induced morphism*
$$\mathrm{GL}(\phi(\mathbf{m})) \times \mathrm{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m}) \to \mathrm{rep}_{Q,I}(\phi(\mathbf{m})), \qquad (g, L) \mapsto g * \Phi(\mathbf{m})(L),$$
is smooth at $(1, 0)$ *for any* $\mathbf{m} \in \mathbb{N}^{(\widetilde{Q})_0}$.
- *The pointed varieties*
$$(\mathrm{rep}_{Q,I}(\phi(\mathbf{m})), N) \qquad \text{and} \qquad (\mathrm{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m}), 0)$$
are smoothly equivalent, where $N := \Phi(\mathbf{m})(0) \simeq \bigoplus X^{m_X}$.

Rank subvariety $\mathcal{C}_M$

- Let $[X, Y] = \dim_{\Bbbk} \mathrm{Hom}_\Lambda(X, Y)$ for any $X, Y \in \mathrm{mod}(\Lambda) \simeq \mathrm{rep}(Q, I)$.

If $M \in \mathrm{rep}_{Q,I}(\mathbf{d})$, then

$$\overline{\mathrm{GL}(\mathbf{d}) * M} \subseteq \{L \in \mathrm{rep}_{Q,I}(\mathbf{d}) \mid [Y, L] \geq [Y, M] \text{ for any indecomposable non projective } Y\}.$$

The right-hand side can be viewed as a (not necessarily reduced !) subvariety of $\mathrm{rep}_{Q,I}(\mathbf{d})$:

- We choose a minimal projective presentation

$$P^1 \xrightarrow{p_Y} P^0 \to Y \to 0$$

  and consider the induced exact sequence

$$0 \to \mathrm{Hom}_\Lambda(Y, L) \to \mathrm{Hom}_\Lambda(P^0, L) \xrightarrow{\mathrm{Hom}_\Lambda(p_Y, L)} \mathrm{Hom}_\Lambda(P^1, L)$$

- $\mathrm{Hom}_\Lambda(p_Y, -)$ can be treated as a morphism

$$\mathrm{rep}_{Q,I}(\mathbf{d}) \to \mathbb{M}_{[S_\mathbf{d}, \tau Y] \times [Y, S_\mathbf{d}]}(\Bbbk),$$

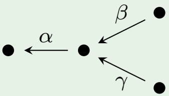
  where $S_\mathbf{d} = \bigoplus (S_i)^{d_i}$ is the standard semisimple representation with dimension vector $\mathbf{d}$.
- We denote by $J^{Q,I,\mathbf{d}}_{Y,r}$ the ideal in $\Bbbk[rep_{Q,I}(\mathbf{d})]$ generated by the images of the minors of size $1 + r$ in $\Bbbk[\mathbb{M}_{[S_\mathbf{d}, \tau Y] \times [Y, S_\mathbf{d}]}(\Bbbk)]$. It does not depend on the chosen minimal presentation.
- Finally,

$$\mathcal{C}_M := \mathrm{Spec}\left(\Bbbk[rep_{Q,I}(\mathbf{d})] \,/\, \sum_Y J^{Q,I,\mathbf{d}}_{Y,[Y,S_\mathbf{d}]-[Y,M]} \right).$$

Example (Dynkin $\mathbb{D}_4$)

Let $\Lambda = \Bbbk Q$, where

Q :



Let $M \in \text{rep}_Q(\mathbf{d})$. The coordinate ring $\Bbbk[\text{rep}_Q(\mathbf{d})] = \Bbbk[x_{i,j}^{\delta}]$, $\delta \in \{\alpha, \beta, \gamma\}$. We arrange the variables $x_{i,j}^{\delta}$ into 3 matrices: X_α, X_β, X_γ. Then the closed subscheme $\mathcal{C}_M \subseteq \text{rep}_Q(\mathbf{d})$ is defined by the ideal generated by minors of appriopriate size of the following 8 matrices:

$$X_\alpha, \quad X_\beta, \quad X_\gamma, \quad X_{\alpha\beta}, \quad X_{\alpha\gamma}, \quad \begin{bmatrix} X_\beta & X_\gamma \end{bmatrix}, \quad \begin{bmatrix} X_{\alpha\beta} & X_{\alpha\gamma} \end{bmatrix}, \quad \begin{bmatrix} X_\beta & X_\gamma \\ X_{\alpha\beta} & 0 \end{bmatrix}.$$

Theorem

Let $\Lambda = \Bbbk Q/I$ and $M \in \text{rep}_{Q,I}(\mathbf{d})$.

- *(Bongartz; Z.) If $\Lambda = \Bbbk Q/I$ is representation finite or tame concealed, then*

$$(\mathcal{C}_M)_{red} = \overline{\text{GL}(\mathbf{d}) * M}.$$

- *(Lakshmibai-Magyar; Riedtmann-Z.) If Q is a Dynkin quiver of type $\mathbb{A}$, then*

$$\mathcal{C}_M = \overline{\text{GL}(\mathbf{d}) * M}.$$

- *Maybe something more for Dynkin quivers of type $\mathbb{D}$ in: [Jiajun Xu, Room 111, today 14:30].*

Rank variety $\mathcal{C}_{M,N}$

- Let Y be a non projective vertex in Γ_Λ. Recall that Y is also a frozen vertex in $\widehat{\Gamma}_\Lambda$ and its unique direct predecessor is denoted by Y'.
- Let $\Omega(\bullet \to Y')$ denote the set of paths in $\widehat{\Gamma}_\Lambda$ from a frozen vertex to Y' passing through non frozen vertices.
- Let $\Omega(Y' \to \bullet)$ denote the set of paths in $\widehat{\Gamma}_\Lambda$ from Y' to a frozen vertex passing through non frozen vertices.
- Observe that the composition $\omega''\omega'$ of a path $\omega' \in \Omega(\bullet \to Y')$ and a path $\omega'' \in \Omega(Y' \to \bullet)$ induces a morphism in $\mathcal{S}_\Lambda(s(\omega'), t(\omega''))$, and also an element in $\Bbbk\widetilde{Q}/\widetilde{I}(t(\omega''), s(\omega'))$.
- We denote by $J_{Y,r}^{\widetilde{Q},\widetilde{I},\mathbf{m}}$ the ideal in $\Bbbk[\mathrm{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m})]$ generated by the images of the minors of size $1+r$ of the (possibly infinite) matrix in regard to the map

$$\mathrm{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m}) \to \mathrm{Hom}_\Bbbk\left(\bigoplus_{\omega''\in\Omega(Y'\to\bullet)} \Bbbk^{m_{t(\omega'')}}, \bigoplus_{\omega'\in\Omega(\bullet\to Y')} \Bbbk^{m_{s(\omega')}}\right), \qquad L \mapsto (L_{\omega''\omega'}).$$

- Let $N := \Phi(\mathbf{m})(0) \simeq \bigoplus X^{m_X}$, $\mathbf{d} := \phi(\mathbf{m}) = \mathbf{dim}\, N$ and choose $M \in \mathrm{rep}_{Q,I}(\mathbf{d})$ such that

$$\mathrm{GL}(\mathbf{d}) * N \subseteq \overline{\mathrm{GL}(\mathbf{d}) * M}.$$

- We define

$$\mathcal{C}_{M,N} := \mathrm{Spec}\left(\Bbbk[rep_{\widetilde{Q},\widetilde{I}}(\mathbf{m})] \,/\, \sum_Y J_{Y,[Y,N]-[Y,M]}^{\widetilde{Q},\widetilde{I},\mathbf{m}}\right).$$

Theorem

Let $\Lambda = \Bbbk Q/I$ *be a standard representation-finite algebra, and* Φ, $\Phi(\mathbf{m})$ *will be as before.*
Let $N := \Phi(\mathbf{m})(0) \simeq \bigoplus X^{m_X}$ *and* $\mathbf{d} := \phi(\mathbf{m}) = \mathbf{dim}\, N$.

- *If* Y *is a non projective vertex in* Γ_Λ *and* $r \geq 0$, *then the ideal* $J_{Y,r}^{\widetilde{Q},\widetilde{I},\mathbf{m}}$ *is generated by the image of* $J_{Y,r+[Y,S_{\mathbf{d}}]-[Y,N]}^{Q,I,\mathbf{d}}$ *via*

$$\Phi(\mathbf{m})^* : \Bbbk[\mathrm{rep}_{Q,I}(\mathbf{d})] \to \Bbbk[\mathrm{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m})].$$

- *Assume that* $M \in \mathrm{rep}_{Q,I}(\mathbf{d})$ *satisfies*

$$\mathrm{GL}(\mathbf{d}) * N \subseteq \overline{\mathrm{GL}(\mathbf{d}) * M}.$$

  Then

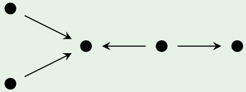
$$\Phi(\mathbf{m})^{-1}(\mathcal{C}_M) = \mathcal{C}_{M,N}.$$

  In particular, the pointed varieties $(\mathcal{C}_M, N)$ *and* $(\mathcal{C}_{M,N}, 0)$ *are smoothly equivalent.*

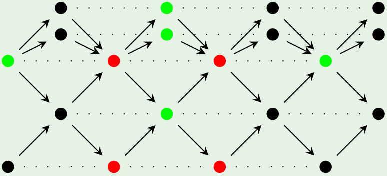
Example (Dynkin $\mathbb{D}_5$)

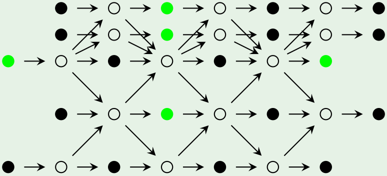
Let $\Lambda = \Bbbk Q$, where

Q :



Then Γ_Λ and $\widehat{\Gamma}_\Lambda$ have the form





The representations $N = \bigoplus$ (green •) and $M = \bigoplus$ (red •) of Q have the same dimension vector $\mathbf{d} = \begin{smallmatrix} 3 & & & \\ & 6 & 4 & 2 \\ 3 & & & \end{smallmatrix}$.
We get a codimension 6 degeneration:

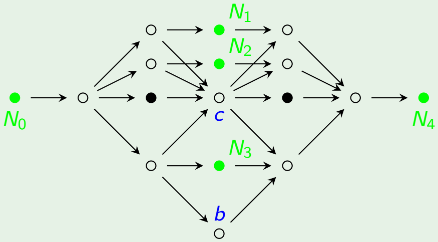
$$\mathrm{GL}(\mathbf{d}) * N \subset \overline{\mathrm{GL}(\mathbf{d}) * M}$$

(the orbits have dimension 60 and 66, respectively). According to the last theorem, $(\mathcal{C}_M, N)$ and $(\mathcal{C}_{M,N}, 0)$ are smoothly equivalent and $\dim \mathcal{C}_{M,N} = 6$.

Example (Dynkin $\mathbb{D}_5$)

We consider the paths in $\widehat{\Gamma}_\Lambda$ between frozen vertices • whose inner vertices are non frozen:

$$x_i\colon N_0 \to \circ \to \circ \to N_i, \qquad y_i\colon N_i \to \circ \to \circ \to N_4, \qquad i \in \{1,2,3\},$$
$$z_{i,j}\colon N_0 \to \circ \to (N_i)' \to c \to (\tau^- N_j)' \to \circ \to N_4, \qquad i,j \in \{1,2,3\},$$
$$t\colon N_0 \to \circ \to \circ \to b \to \circ \to \circ \to N_4.$$

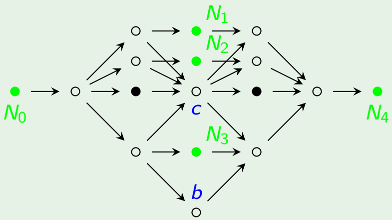


Let $\mathbf{m} = \mathbf{dim}(S_{N_0} \oplus \cdots \oplus S_{N_4})$. Then $\mathrm{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m}) \simeq \mathrm{Spec}\left(\Bbbk[x_i, y_i, z_{i,j}, t]/J\right)$, where J is generated by

$$\sum z_{i,1}, \sum z_{i,2}, \sum z_{i,3}, \quad x_1y_1 + z_{1,1}, \quad x_2y_2 + z_{2,2}, \quad x_3y_3 + z_{3,3} + t, \quad \sum z_{1,j}, \sum z_{2,j}, \sum z_{3,j}.$$

$\mathcal{C}_{M,N} \subset \mathrm{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m})$ is given by the ideal generated by t and the minors of size 2 of the matrix $[z_{i,j}]$.

Example (Dynkin $\mathbb{D}_5$)



Let $\mathbf{m} = \mathbf{dim}(S_{N_0} \oplus \cdots \oplus S_{N_4})$. Then $\mathrm{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m}) \simeq \mathrm{Spec}\left(\Bbbk[x_i, y_i, z_{i,j}, t]/J\right)$, where J is generated by

$$\sum z_{i,1}, \sum z_{i,2}, \sum z_{i,3}, \quad x_1y_1 + z_{1,1}, \quad x_2y_2 + z_{2,2}, \quad x_3y_3 + z_{3,3} + t, \quad \sum z_{1,j}, \sum z_{2,j}, \sum z_{3,j}.$$

$\mathcal{C}_{M,N} \subset \mathrm{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m})$ is given by the ideal generated by t and the minors of size 2 of the matrix $[z_{i,j}]$. Hence

$$\mathcal{C}_{M,N} \simeq \mathrm{Spec}\Big(\Bbbk[x_i, y_i, z_{1,2}]/\ (\ (z_{1,2})^2 + z_{1,2} \cdot (-x_1y_1 - x_2y_2 + x_3y_3) + x_1y_1x_2y_2\ )\ \Big)$$

is a 6-dimensional hypersurface. Since it is reduced, $(\overline{\mathrm{GL}(\mathbf{d}) * M}, N)$ and $(\mathcal{C}_{M,N}, 0)$ are smoothly equivalent.

Potential applications of the main theorems

- Describing types of singularities of orbit closures in codimension 2 (using algebraic geometry methods for surfaces).
- Describing types of generic singularities.
- Describing tangent spaces and singular locus of orbit closures for directed algebras.
- Finding examples when $\mathcal{C}_M$ is not reduced for directed algebras (if such examples exist).

THANK YOU !