n-torsionfree objects and Frobenius Functors

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1. Introduction and Motivations

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A module is Gorenstein projective if and only if it is a module of G-dimension zero when it is a **finitely generated module over a two-sided Noetherian ring**. As the origin of Gorenstein homological algebra, the notion of modules of G-dimension zero was defined in terms of n-torsionfreeness in [AB].

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[AB] M. Auslander and M. Bridger, Stable module theory, Memoirs of the Amer. Math. Soc. 94(1969).

[R] W. Ren, Gorenstein projective and injective dimensions over Frobenius extensions, Comm. Algebra 46(2008), 1-7.

[Z1] Z.-B. Zhao, Gorenstein homological invariant properties under Frobenius extensions, Sci. China Math. 62(2019), 2487-2496.

1. Introduction and Motivations

Recall: Let R be a two-sided Noetherian ring. For a module $M \in R$-mod, there is a projective resolution $P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$ in R-mod.

Applying by the functor $(-)^* = \mathrm{Hom}_R(-, R)$, we have an exact sequence

$$0 \longrightarrow M^* \longrightarrow P_0^* \xrightarrow{f^*} P_1^* \longrightarrow \mathrm{Coker} f^* \longrightarrow 0 ,$$

and call $\mathrm{Coker} f^*$ the transpose of M, denote it by $\mathrm{Tr} M$.

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An R-module M is said to be n**-torsionfree** if $\mathrm{Ext}^i_{R^{op}}(\mathrm{Tr} M, R) = 0$ for $1 \leq i \leq n$.

If an R-module M is n-torsionfree for any positive integer n, then it is called an ∞**-torsionfree** module.

1. Introduction and Motivations

A module M in R-mod is said to be of **G-dimension zero**, denoted it by $\text{G-dim}_R(M) = 0$, if it is satisfies: (1) M is reflexive; (2) $\text{Ext}^i_R(M, R) = 0$ for all $i > 0$; (3) $\text{Ext}^i_{R^{\text{op}}}(M^*, R) = 0$ for all $i > 0$.

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By definition of n-torsionfreeness and the fact that $\text{Tr}(\text{Tr}M) \cong M$, we have, for an R-module M, $\text{G-dim}_R(M) = 0$ if and only if $M \in {}^{\perp}R$ and M is ∞-torsionfree if and only if M and $\text{Tr}M$ (as a right R-module) are both ∞-torsionfree.

1. Introduction and Motivations

In [Z], Zhao obtain that n-torsionfreeness of modules is preserved under Frobenius extensions.

Theorem

([Z] Theorem 3.5) *Let R and S be two-sided Noetherian rings and $l : R \to S$ be a Frobenius extension. Let M be an S-module and n a positive integer. Then M is n-torsionfree as an S-module if and only if M is n-torsionfree as an R-module.*

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Question: What is happened in R-Mod over arbitrary ring?

[Z] Z.-B. Zhao, k-torsionfree modules and Frobenius extensions, J. Algebra 624 (2024), 49-65.

1. Introduction and Motivations

Recall: Let $\mathcal{C}$ be a full subcategory of an Abelian category $\mathcal{A}$ and $C \in \mathcal{C}$, $A \in \mathcal{A}$. An $\mathcal{A}$-homomorphism $A \to C$ is said to be a **left $\mathcal{C}$-approximation** of A if $\mathrm{Hom}_{\mathcal{A}}(C, X) \to \mathrm{Hom}_{\mathcal{A}}(M, X)$ is epic for any $X \in \mathcal{C}$. A subcategory $\mathcal{C}$ is said to be **covariantly finite** in $\mathcal{A}$ if every object in $\mathcal{A}$ has a left $\mathcal{C}$-approximation.

1. Introduction and Motivations

Let $\mathcal{A}$ be an abelian category with enough projective objects. The latter condition means that for each object M, there is an epimorphism $P \to M$ with P projective.

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Definition

Let $n \geq 0$ be an integer. An object M in $\mathcal{A}$ is said to be n**-torsionfree** provided that there exists an exact sequence $0 \longrightarrow M \xrightarrow{f_1} P_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} P_n$ with $P_i \in \mathcal{P}(\mathcal{A})$, such that each $\mathrm{Im} f_i \to P_i$ is a left $\mathcal{P}(\mathcal{A})$-approximation of $\mathrm{Im} f_i$ for $1 \leq i \leq n$. We denote the full subcategory of $\mathcal{A}$ consisting of all n-torsionfree objects by $\mathcal{T}^n(\mathcal{A})$.

An object M is called an ∞**-torsionfree** if it is n-torsionfree for any positive integer n.

Remark: Let the abelian category $\mathcal{A}$ be the left finitely generated R-module category R-mod with R a two-sided Noetherian ring. Then an object M which is n-torsionfree in $\mathcal{A}$ is just an n-torsionfree module; see [AB].

[AB] M. Auslander and M. Bridger, Stable module theory, Memoirs of the Amer. Math. Soc. 94(1969).

1. Introduction and Motivations

Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be two additive functors. We say that (F, G) is a **Frbenius pair** between $\mathcal{A}$ and $\mathcal{B}$, provided that both (F, G) and (G, F) are adjoint pairs; see [M65] or [CGN]. We call the functor F a **Frobenius functor**, if it fits into a Frobenius pair (F, G). In this case, the functor G is also a Frobenius functor. In other words, Frobenius functors always appear in pairs.

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[CGN] F. Castaño Iglesias, J. Gómez Torrecillas, and C. Năstăsescu, Frobenius functors:applications, Comm. Algebra 27 (10)(1999) 4879-4900.

[M65] K. Morita, Adjoint pairs of functors and Frobenius extensions, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A 9 (1965) 40-71.

1. Introduction and Motivations

Remark: By Theorem 1.2 in [Kad], a ring extension (or a ring homomorphism) $l : R \to S$ is Frobenius if and only if $({}_SS \otimes_R -(\cong \mathrm{Hom}_R({}_RS_S, -)), Res)$ is a Frobenius pair between R-Mod and S-Mod. We refer to [Kad] for more details, and the examples can found in Example 2.4 in [Z1].

[Kad] L. Kadison, New examples of Frobenius extensions, University Lecture Series, Vol 14, AMS. Provedence, Rhode Island, 1999.

[Z1] Z.-B. Zhao, Gorenstein homological invariant properties under Frobenius extensions, Sci. China Math. 62(2019), 2487-2496.

1. Introduction and Motivations

Setup and notation. Throughout, we assume that both $\mathcal{A}$ and $\mathcal{B}$ are abelian categories with enough projective objects. Denote by $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{B})$ the full subcategories of projective objects of $\mathcal{A}$ and $\mathcal{B}$, respectively.

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Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor with a right adjoint $G : \mathcal{B} \to \mathcal{A}$. we will denote the adjoint pair (F, G) by $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$.

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Aim: We will give some properties of n-torsionfree objects and investigate the transfer of n-torsionfreeness under Frobenius functors in this note.

2. n-torsionfree objects

An object M in $\mathcal{A}$ is said to be n-**torsionfree** provided that there exists an exact sequence $0 \longrightarrow M \xrightarrow{f_1} P_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} P_n$ with $P_i \in \mathcal{P}(\mathcal{A})$, such that each $\mathrm{Im} f_i \to P_i$ is a left $\mathcal{P}(\mathcal{A})$-approximation of $\mathrm{Im} f_i$ for $1 \le i \le n$.

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Remark: Every object in $\mathcal{A}$ is 0-torsionfree and every projective object is ∞-torsionfree. An object M is m-torsionfree must be n-torsionfree when $m > n$. And we have a decreasing sequences of subcategories of $\mathcal{A}$

$$\mathcal{T}^1(\mathcal{A}) \supseteq \mathcal{T}^2(\mathcal{A}) \supseteq \cdots \supseteq \mathcal{T}^n(\mathcal{A}) \supseteq \cdots .$$

2. n-torsionfree objects

Proposition 2.1. Let $\mathcal{A}$ be an abelian category with enough projective objects and n be a positive integer. The subcategory $\mathcal{T}^n(\mathcal{A})$ is closed under direct sums and summands.

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The following observation gives a recurrence relation of n-torsionfr -eeness.

Proposition 2.2. Let $0 \longrightarrow M \xrightarrow{f} P \longrightarrow N \longrightarrow 0$ be a short exact sequence in $\mathcal{A}$ with f a left $\mathcal{P}(\mathcal{A})$-approximation of M and let $n \geq 1$ be an integer. Then M is n-torsionfree if and only if N is $(n-1)$-torsionfree.

2. n-torsionfree objects

Analogy to the notion of classical the module with G-dimension zero.

Definition

Let $\mathcal{A}$ be an abelian category with enough projective objects. An object M is said to be of **G-dimension zero**, denoted it by $\text{G-dim}(M) = 0$, if it is satisfies: (1) $\operatorname{Ext}^i_{\mathcal{A}}(M, \mathcal{P}(\mathcal{A})) = 0$ for all $i > 0$; (2) M is ∞-torsionfree.

2. n-torsionfree objects

Recall that an acyclic complex

$$P^\bullet = \quad \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective objects is said to be **totally acyclic**, provided it remains acyclic after applying $\mathrm{Hom}_{\mathcal{A}}(-, P)$ for any projective object $P \in \mathcal{A}$. An object $M \in \mathcal{A}$ is called **Gorenstein projective** if there is a totally acyclic complex $P^\bullet$ such that M is isomorphic to its zeroth cocycle $Z^0(P^\bullet)$.

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We use $\mathcal{GP}(\mathcal{A})$ to denote the subcategory consisting of all Gorenstein projective objects of $\mathcal{A}$.

Proposition 2.3. Let $\mathcal{A}$ be an abelian category with enough projective objects and M be an object in $\mathcal{A}$. Then M is Gorenstein projective if and only if M is of G-dimension zero.

3. Weakly Gorenstein categories

As a generalization of notion Gorenstein algebras, Ringel and Zhang in [RZ] introduced that of weakly Gorenstein algebra.

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Definition

Let $\mathcal{A}$ be an abelian category with enough projective objects. We call that $\mathcal{A}$ is **weakly Gorenstein** if $\mathcal{GP}(\mathcal{A}) = {}^{\perp}\mathcal{P}(\mathcal{A})$, where ${}^{\perp}\mathcal{P}(\mathcal{A}) = \{X \mid \mathrm{Ext}^i_{\mathcal{A}}(X, \mathcal{P}(\mathcal{A})) = 0 \text{ for } i \geq 1\}$.

An artin algebra A is said to be left (resp. right) weakly Gorenstein if the category of all left (resp. right) finitely generated A-modules is weakly Gorenstein.

[RZ] C.M.Ringel and P. Zhang, Gorenstein-projective modules and semi-Gorenstein-projective modules, Algebra Number Theory 14(2020) 1-36.

3. Weakly Gorenstein categories

The following is analogy to Theorem 1.2 in [RZ].

Proposition 3.1. Let $\mathcal{A}$ be an abelian category with enough projective objects. The following statements are equivalent.

(1) The category $\mathcal{A}$ is weakly Gorenstein.

(2) Every object in ${}^{\perp}\mathcal{P}(\mathcal{A})$ is ∞-torsionfree.

(3) Every object in ${}^{\perp}\mathcal{P}(\mathcal{A})$ is n-torsionfee, where n is a positive integer.

3. Weakly Gorenstein categories

Proposition 3.2. Let $\mathcal{A}$ be an abelian category with enough projective objects. If $\mathcal{T}^n(\mathcal{A}) = \mathcal{T}^{n+1}(\mathcal{A})$ for some positive integer n, then $\mathcal{A}$ is weakly Gorenstein.

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Corollary 3.3. Let $\mathcal{A}$ be an abelian category with enough projective objects. If $\mathcal{A}$ has only finitely many isomorphism classes of n-torsionfree objects for some positive integer n, then $\mathcal{A}$ is weakly Gorenstein.

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Remark: Recall that an artin algebra A is said to be **torsionless-finite** if there are only finitely many isomorphism classes of indecomposable torsionless left A-modules.

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In [RZ], Ringel and Zhang show that any torsionless-finite artin algebra is left weakly Gorenstein; see ([RZ] 3.6). The corollary above implies that if an artin algebra A has only finitely many isomorphism classes of indecomposable n-torsionfree modules for some n, then A is left weakly Gorenstein.

4. Frobenius Functors

In this section, we will investigate the transfer of n-torsionfreeness under Frobenius functors.

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Proposition 4.1. Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be a Frobenius pair between two abelian categories with enough projective objects. Assume that F is faithful and X be an object in $\mathcal{A}$. Then we have $X \in {}^{\perp}\mathcal{P}(\mathcal{A})$ if and only if $F(X) \in {}^{\perp}\mathcal{P}(\mathcal{B})$.

4. Frobenius Functors

Theorem 4.2. Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be a Frobenius pair between two abelian categories with enough projective objects. Assume that F is faithful and X be an object in $\mathcal{A}$. Then X is n-torsionfree in $\mathcal{A}$ if and only if so is $F(X)$ in $\mathcal{B}$.

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Remark: The faithful condition is necessary. Take another abelian category $\mathcal{A}'$ with enough projective objects. An object (X, X') in the product category $\mathcal{A} \times \mathcal{A}'$ is n-torsionfree if and only if so are X and X'.

Consider the canonical projection functor $Pr : \mathcal{A} \times \mathcal{A}' \to \mathcal{A}$ and inclusion functor $Inc : \mathcal{A} \to \mathcal{A} \times \mathcal{A}'$. Then (Pr, Inc) is a Frobenius pair. It is clear that X is n-torsionfree in $\mathcal{A}$ can not induce that (X, X') is n-torsionfree in the product category $\mathcal{A} \times \mathcal{A}'$. Hence, the Frobenius functor Pr does not reflect n-torsionfree objects in general.

4. Frobenius Functors

Example: A Frobenius extension $l : R \to S$ yields a classical Frobenius pair $(S \otimes_R -, Res)$ between R-Mod and S-Mod. It is clear that the restriction functor Res is faithful. It follows from Theorem 4.2 that a left S-module ${}_SM$ is n-torsionfree if and only if the underlying R-module ${}_RM$ is n-torsionfree. This result is due to Theorem 3.5 in [Z].

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In a Frobenius extension $l : R \to S$, we assume further that S is progenerator as a right R-module. Then the Frobenius functor $S \otimes_R - : R\text{-Mod} \to S\text{-Mod}$ is faithful. We apply Theorem 4.2 to obtain that a left R-module ${}_RM$ is n-torsionfree if and only if so is the left S-module ${}_SS \otimes_R M$.

Corollary 4.3. Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be a Frobenius pair between two abelian categories with enough projective objects. Assume that F is faithful and X be an object in $\mathcal{A}$. Then X is ∞-torsionfree in $\mathcal{A}$ if and only if so is $F(X)$ in $\mathcal{B}$.

Corollary 4.3. Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be a Frobenius pair between two abelian categories with enough projective objects. Assume that F is faithful and X be an object in $\mathcal{A}$. Then X is ∞-torsionfree in $\mathcal{A}$ if and only if so is $F(X)$ in $\mathcal{B}$.

Combining Proposition 4.1 and Corollary 4.3, we have the following well-known result.

Proposition 4.4.([CR] Theorem 3.2) Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be a Frobenius pair between two abelian categories with enough projective objects. Assume that F is faithful and M is an object in $\mathcal{A}$. Then M is Gorenstein projective (or M is of G-dimension zero) in $\mathcal{A}$ if and only if so is $F(M)$ in $\mathcal{B}$.

[CR] X. -W. Chen and W. Ren, Frobenius functors and Gorenstein homological properties, J. Algebra 610 (2022), 18-37.

4. Frobenius Functors

A two-sided Noetherian ring R is called *left (right) quasi n-Gorenstein* if the left (right) flat dimension of the i-st term in a minimal injective resolution of R as a left (right) R-module is less than or equal to i for any $1 \leq i \leq n$; see [H2].

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Corollary 4.5. Let $l : R \to S$ be a Frobenius extension with S a generator as a right R-module and n a positive integer. Then R is a left (resp. right) quasi n-Gorenstein ring if and only if so is S.

[H2] Z. -Y. Huang, ω^t-approximation representations over quasi k-Gorenstein algebras, Sci. China (Series A), 42 (1999), 945-956.

4. Frobenius Functors

A ring extension $l : R \to S$ is called **excellent extensions** if it satisfies: (1) S is left R-projective, that is, if ${}_SM$ is a module and ${}_SN$ is a submodule then ${}_RN|{}_RM$ implies that ${}_SN|{}_SM$, where $N|M$ means that N is a summand of M. (2) S is a **free normal** extension of R, i.e. $S = \sum_{i=1}^{n} s_iR$ and S is free with common basis $\{s_1 = 1, s_2, \cdots, s_n\}$ as both a left R-module and a right R-module, such that $s_iR = Rs_i$ for $1 \leq i \leq n$; see [P]. The examples of excellent extensions can be found in Example 2.2 in [HS].

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Corollary 4.6 Let $\theta : R \to S$ be an excellent extension. Then R is a left (resp. right) quasi n-Gorenstein ring if and only if so is S.

[HS] Z. -Y. Huang and J. -X. Sun, Invariant properties of reprenstations under excellent extensions, J. Algebra, 358 (2012), 87-101.

4. Frobenius Functors

A two-sided Noetherian ring is said to satisfy the **Auslander condition** if the flat dimension of the i-th term of minimal injective coresolution of R as an right R-module at most $i-1$ for any $i \geq 1$; see [FGR].

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A known result in theory of commutative ring states that a commutative Noetherian ring is Gorenstein if and only if it satisfies the Auslander condition. Based on it, Auslander and Reiten [AR] conjectured that an Artin algebra satisfying the Auslander condition is Gorenstein. We call this conjecture **ARC** for short.

[FGR] R. M. Fossum, P. A. Griffith and I. Reiten, Trivial Extensions of Abelian Categories, in: Homological Algebra of Trivial Extensions of Abelian Categories with Applications to Ring Theory, in: Lecture Notes in Math., vol. 456, Springer-Verlag, Berlin, New York, 1975.

[AR] M. Auslander and I. Reiten, k-Gorenstein algebras and syzygy modules, J. Pure Appl. Algebra 92 (1994), 1-27.

In [H], Huang proved the following.

Theorem 4.7.([H] Corollary 4.12) Let R be an Artin algeba. If R satisfies the Auslander condition, then the following statements are equivalent.

(1) R is Gorenstein.

(2) R is left or right weakly Gorenstein.

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(2) R is left or right weakly Gorenstein.
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[H] Z.-Y. Huang, Auslander-type conditions and weakly Gorenstein algebras, Preprint, 2024.

Proposition 4.8. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two abelian categories with enough projective objects. Let F be a faithful Frobenius functor from $\mathcal{A}$ to $\mathcal{B}$. If $\mathcal{B}$ is weakly Gorenstein, then so is $\mathcal{A}$.

Proposition 4.8. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two abelian categories with enough projective objects. Let F be a faithful Frobenius functor from $\mathcal{A}$ to $\mathcal{B}$. If $\mathcal{B}$ is weakly Gorenstein, then so is $\mathcal{A}$.

Corollary 4.9. Let $l : R \to S$ be a Frobenius extension. If the base ring R is weakly Gorenstein, then so is S.

4. Frobenius Functors

By Theorem 4.7 above and Corollary 4.9, we have

Corollary 4.10. Let R and S be two Artin algebras and $l : R \to S$ be a Frobenius extension. If **ARC** holds true for the base ring R, then **ARC** also holds true for the extension ring S.

Corollary 4.11. Let R and S be two Artin algebras and $l : R \to S$ be an excellent extension. Then **ARC** holds true for the base ring R if and only if **ARC** holds true for the extension ring S.

Thank you
for your attention!!