A modular construction of Positroid varieties

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Affine permutations

Definition (Bounded affine permutation)

Given integers, $0 < k < n$, a bijection $f : \mathbb{Z} \to \mathbb{Z}$ is called a bounded affine permutation of period n and average k if

1. $f(i+n) = f(i) + n$,
2. $i \leq f(i) \leq i + n$
3. $\sum_{i=1}^{n} f(i) - i = k \cdot n$.

We denote by $B(k,n)$ the set of such permutations.

Example

Let M be a full rank $k \times n$ matrix and $v_1, \ldots, v_n$ be the column vectors of M. Set

$$f_M(i) = \min\{j \geq i \,\big|\, v_i \in \mathrm{span}(v_{i+1}, \ldots, v_j)\}$$

where the index are modulo n. One can check that $f_M \in B(k,n)$.

Positroid varieties

Definition (Positroid varieties)

Given $f \in B(k,n)$, the *(open) positroid variety of type f* is defined to be

$$\Pi_f = GL_k \Big\backslash \{M \in Mat_{k\times n} \Big| f_M = f\},$$

where GL_k acts by row transformations.

Remark (Properties of Π_f)

1. *Π_f is nonempty and $G(k,n) = \bigsqcup_{f\in B(k,n)} \Pi_f$,*
2. *Π_f is smooth, affine and irreducible,*
3. *$\overline{\Pi_f} = \bigsqcup_{g\geq f} \Pi_g$ and $\overline{\Pi_f}$ has CM singularities,*
4. *Elements of $B(k,n)$ is in one to one correspondence with pairs (v,w) where v is a grassmannian permutation and $w \leq v$.*

Poisson structure on moduli of complexes

Let Y be a projective variety. Denote by $\mathbf{Cplx}(Y)$ the moduli stack of bounded complexes of vector bundles on Y up to *chain isomorphisms*. Denote by $\mathbf{Vect}^{\mathbb{Z}}(Y)$ the stack of $\mathbb{Z}$-graded vector bundles and by $\mathbf{Perf}(Y)$ the stack of perfect complexes.

Definition (Calabi-Yau varieties)

A projective variety Y is called *nCY* if it is a Gorenstein n-fold such that $\omega_Y \cong \mathcal{O}_Y$.

Theorem (Pantev-Toen-Vaquie-Vezzosi)

Let Y be a nCY variety. The moduli stack $\mathbf{Perf}(Y)$ (resp. $\mathbf{Vect}^{\mathbb{Z}}(Y)$) admits a $(2-n)$-shifted symplectic structure.

Example (CY curves)

When $n = 1$, all CY curves (with planar singularities) are classified by Kodaira. Semi-stable ones are precisely smooth elliptic curves and Kodaira cycles (Neron polygons).

Poisson structure on moduli of complexes

Consider the roof diagram

$$\mathbf{Vect}^{\mathbb{Z}}(Y) \xleftarrow{\;p\;} \mathbf{Cplx}(Y) \xrightarrow{\;q\;} \mathbf{Perf}(Y)$$

where p forgets the differential of the complex and q sends the chain complex to the underlying object in the derived category.

Theorem (H-Polishchuk)

Let Y be a nCY variety. The above diagram is a Lagrangian correspondence with respect to the $(2-n)$-shifted symplectic structure of PTVV. And $\mathbf{Cplx}(Y)$ is equipped with a $(1-n)$-shifted Poisson structure.

Theorem (H-Polishchuk)

When $n = 1$, the fibers of p (resp. of q) are Poisson and the fibers of (p, q) are symplectic.

Grassmannian with Feigin-Odesskii Poisson structure

Let C be a CY curve and ξ be a simple vector bundle of degree n on C. For $0 < k < n$, let $\mathcal{N}_{\xi,k}$ be the moduli stack of complexes $\mathcal{O}^k \to E$ with fixed quotient $E/\mathcal{O}^k \cong \xi$ and a stability condition. One can show that $\mathcal{N}_{\xi,k}$ is a $\mathbb{G}_m$-gerbe over $G(k,n)$.

Corollary

Given a pair (C,F) of CY curve and a simple bundle of degree n on C, we get a Poisson structure $\pi_{C,F}$ on $G(k,n)$. The sub moduli space of complexes $\mathcal{O}^k \to E$, where E is fixed up to isomorphisms, is symplectic.

Example (Feigin-Odesskii elliptic Poisson structure)

When C is a smooth elliptic curve, ξ is a simple bundle and $k = 1$, we get the Feigin-Odesskii Poisson structure on $\mathbb{P}\mathrm{Ext}^1(\xi, \mathcal{O})$.

We will call $\pi_{C,F}$ the *(generalized) Feigin-Odesskii Poisson structure*.

Positroid varieties as T-Poisson subvariety

Let $C = C^n$ be the Kodaira cycle with n irreducible components, i.e. the Neron n-gon. And L be a line bundle with degree vector $(1, \ldots, 1)$. Denote by $T \subset \mathbb{G}_m^n \subset Aut(C^n)$ the sub torus defined by $t_1 t_2 \ldots t_n = 1$. Since L is T-invariant, the morphism

$$p : \mathcal{N}_{L,k} \to \mathbf{Vect}^L(C)$$

is T-equivariant. Here $\mathbf{Vect}^L(C)$ is the stack of vector bundles with determinant L and p sends $\mathcal{O}^k \to E$ to E.

Theorem (H-Polishchuk)

The Feigin-Odesskii poisson structure $\pi_{C^n,L}$ is T-invariant. As a consequence, the p-preimage of a T orbit in $\mathbf{Vect}^L(C)$ is a T-Poisson variety.

Symplectic fibration on Positroid varieties

Let $E = \bigoplus_{i=1}^{p} E_i$ such that E_i are indecomposables. T-action preserves the *numerical invariants* of E_i.

Theorem (H-Polishchuk)

Let E be a vector bundle in the image of $p : \mathcal{N}_{L,k} \to \mathbf{Vect}^L(C)$.

1. *The T-orbits of E are in one to one correspondence with elements of $B(k, n)$.*
2. *Given $f \in B(k, n)$, denote by E_f the T-orbits in $\mathbf{Vect}^L(C)$ that corresponds to f. The adjacency order coincides with the Bruhat order on $B(k, n)$.*
3. $p^{-1}(E_f) \cong \Pi_f$.
4. *The restricted map $p : p^{-1}(E_f) \to E_f$ is a smooth morphism with connected and symplectic fibers.*

Remark

$\pi_{C^n,L}$ coincides with the induced Poisson structure by the Drinfeld-Jimbo trignometric r-matrix on GL_n.

Generalized Positroid varieties

Define $f_{n,k} \in B(k,n)$ by $f_{n,k}(i) = i + k$ for $i \in \mathbb{Z}$. It is the unique minimal element in $B(k,n)$. We call $D = \bigsqcup_{g > f_{n,k}} \Pi_g$ the *degeneracy divisor*. Note that D does not have simple normal crossing singularities. Our modular construction gives several generalizations of positroid varieties.

1. When $C = C^n$ is a Kodaira cycle and $L(\mathbf{d})$ is a degree n line bundle with degree vector $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n$. We get a codimension one torus $T(\mathbf{d}) \subset \mathbb{G}_m^n \subset Aut(C^n)$ such that the poisson structure $\pi_{C,^n L(\mathbf{d})}$ is $T(\mathbf{d})$-invariant.
2. The previous poisson structure can be further generalized to $\pi_{C^n,F}$ where F is a simple vector bundle with determinant $L(\mathbf{d})$. Similarly, there exists a codimension one torus T_F such that $\pi_{C^n,F}$ is T_F-invariant.
3. When C is chosen to be other Kodaira fibers, the Poisson structure might still admit a torus symmetry but of smaller dimension, e.g. C being a cuspidal curve.

The T-Poisson vareities appear in these general context are called *generalized positroid varieties*.

Some open problems

Let C be a CY curve and F be a simple vector bundle of degree n. For $0 < k < n$, recall that the moduli stack $\mathcal{N}_{F,k}$ is a $\mathbb{G}_m$-gerbe over $G(k,n)$, equpipped with Poisson structure $\pi_{C,F}$. Denote by Π_0 the sub moduli stack of complex $\mathcal{O}^k \to E$ where

- E simple when $\gcd(n, k + \mathrm{rk}(F)) = 1$,
- E is semi-simple when $\gcd(n, k + \mathrm{rk}(F)) > 1$.

Question (Work in progress)

Is Π_0 log Calabi-Yau in the sense of Gross-Hacon-Keel? If yes, are all minimal models "modular"?

Question

When $C = C^n$ is a Neron polygon, does Π_0 admit a cluster algebra structure?

Thank you!