Simple modules and Quasi-tubes on self-injective algebras of polynomial growth

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Contents

Background

- Koenig and Liu [KL12] introduced simple-minded systems (sms's for short) in the stable module category of any artin algebra. They showed that the cardinality of an sms is finite and sms's are invariant under stable equivalences.

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- Koenig and Liu [KL12] introduced simple-minded systems (sms's for short) in the stable module category of any artin algebra. They showed that the cardinality of an sms is finite and sms's are invariant under stable equivalences.
- Malicki and Skowroński [MS11] showed that, for a self-injective algebra, the number of simple modules and projective modules lying in a quasi-tube $\mathcal{C}$ is less than or equal to $r(\mathcal{C}) - 1$.
- Chan, Liu and Zhang [CLZ20] showed that the number of elements in an sms of self-injective algebra A lying in a quasi-tube $\mathcal{C}$ is strictly less than $r(\mathcal{C})$ the rank of $\mathcal{C}$.

Notations

A: finite dimensional algebra over algebraically closed field k.
A-mod: the category of finite dimensional left A-modules.
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Ω^{-1}: the cosyzygy functor which assigns to any object M of A-mod the kerner of its projective cover $P_A(M) \twoheadrightarrow M$ in A-mod.
$r(\mathcal{C})$: the rank of a quasi-tube $\mathcal{C}$.
$\mathcal{Q}_n$: the family of quasi-tubes of rank $n(\geq 1)$.

Notations

Let A be a self-injective algebra. For any families $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}$ of objects in A-$\underline{mod}$, define

$$\mathcal{S}_1 * \mathcal{S}_2 := \{X \in A\text{-}\underline{mod} \mid \text{ there is a trigngle } M \to X \to N \to M[1],$$

$$M \in \mathcal{S}_1, N \in \mathcal{S}_2\}.$$

Take $(\mathcal{S})_0 := \{0\}$ and inductively define $(\mathcal{S})_n := (\mathcal{S})_{n-1} * (\mathcal{S} \cup \{0\})$ for any positive integer n. $\mathcal{S}$ is called *extension-closed*, if $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$.
$\mathcal{F}(\mathcal{S}) := \bigcup\limits_{n \in \mathbb{N}} (\mathcal{S})_n$: the *extension closure* of $\mathcal{S}$, which is the smallest extension closed full subcategory containing $\mathcal{S}$ in A-$\underline{mod}$.

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- Note that stable module category $A\text{-}\underline{mod}$ is a triangulated category with suspension functor Ω^{-1} (denoted by [1]), and its triangles are induced by short exact sequences in $A\text{-}mod$.

Definitions

Definition 1 ([D15, KL12])

Let A be a self-injective k-algebra. A family of objects $\mathcal{S}$ in A-$\underline{mod}$ is called a **simple-minded system** (sms for short) if

1. (Orthogonality) $\mathcal{S}$ is an orthogonal system, that is, for any $M, N \in \mathcal{S}$, $\underline{\mathrm{Hom}}_A(M, N) \cong \begin{cases} k & (M = N), \\ 0 & (M \neq N). \end{cases}$
2. (Generating condition) $\mathcal{F}(\mathcal{S}) = A$-$\underline{mod}$.

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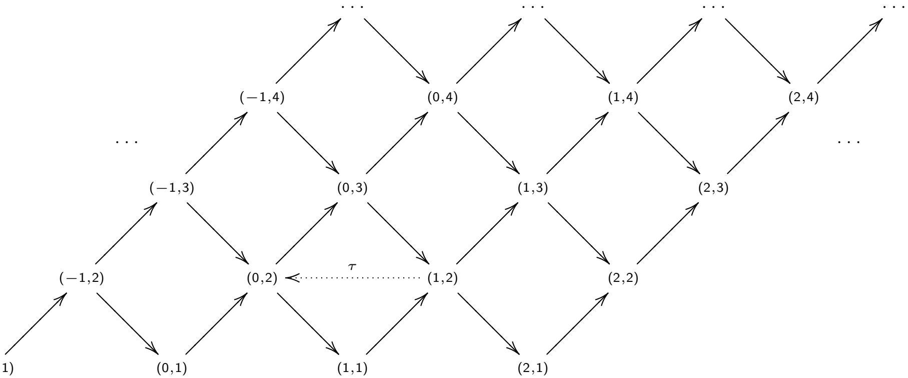
Definition 2

Let A be a k-algebra and $\mathcal{C}$ a connected component of the AR-quiver of A. $\mathcal{C}$ is called a **quasi-tube of rank** $n \geq 1$, if the stable part (by removing projective modules) of $\mathcal{C}$ is of the form $\mathbb{Z}A_\infty/\langle \tau^n \rangle$(see $\mathbb{Z}A_\infty$ in the following diagram).

Definitions



Question and main result

Observation:
There are local algebras whose simple module is contained in a quasi-tube. For example, the algebra $k[x, y]/ < x^2, y^2, xy - yx >$ did. In general, It holds for the local algebras of quaternion type whose AR-component consists of quasi-tubes of rank ≤ 2.

Question and main result

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But many non-local algebras do not satisfy this property.

Theorem 3 (Main result)

Let A be a non-local self-injective algebra of polynomial growth. Then the set of simple modules is not contained in a quasi-tube.

Self-injective algebras of polynomial growth

According to Białkowski, Erdmann and Skowroński, the following theorem summarizes the structure of a self-injective algebra of polynomial growth.

Theorem 4 ([S06])

Let A be a nonsimple, basic, indecomposable self-injective algebra. Then

1. *A is a standard algebra of polynomial growth if and only if A is isomorphic to a self-injective algebra of Dynkin type, Euclidean type or tubular type.*
2. *A is of polynomial growth if and only if A is socle equivalent to a standard self-injective algebra of polynomial growth.*

We consider two cases as follows: one is the Dynkin type and Euclidean type case, the other is the tubular type case.

The Dynkin type and Euclidean type case

- Since the algebras which are socle equivalent to the Dynkin type are of finite representation type, they do not have quasi-tubes. Thus our conclusion holds for the Dynkin type case naturally.

Lemma 5 (Key lemma)

Let A be a self-injective algebra and $\mathcal{S}$ an sms in A-$\underline{\text{mod}}$. If there is a non-projective, non-τ-periodic module in A-$\underline{\text{mod}}$, then $\mathcal{S}$ contains at least one non-τ-periodic module.

The Dynkin type and Euclidean type case

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Corollary 6

Let A be a non-periodic self-injective algebra. Then the set of simple A-modules is not contained in a quasi-tube.

The tubular type case

Definition 7 ([BS02])

A **self-injective algebra of tubular type** is an algebra of the form $\widehat{B}/G$, where B is a tubular algebra and G is an admissible group of automorphisms of the repetitive algebra $\widehat{B}$ of B.

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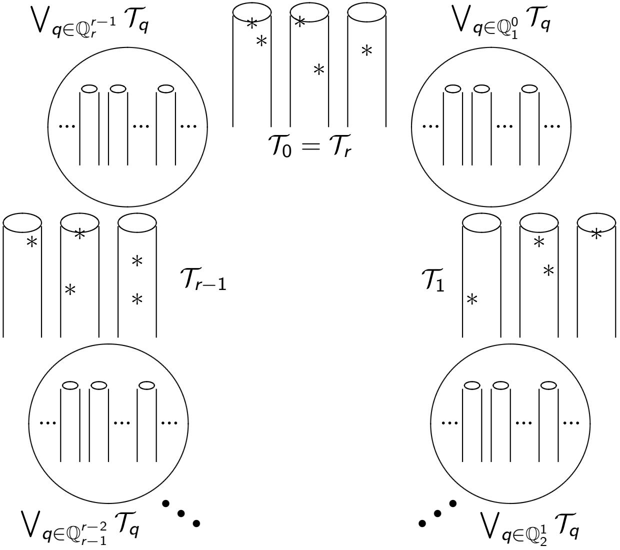
Self-injective algebra of tubular type is fully classified by Białkowski, Lenzing and Skowroński [BS02, LS00], including

- Self-injective algebras of tubular type $(2, 2, 2, 2)$,
- Self-injective algebras of tubular type $(3, 3, 3)$,
- Self-injective algebras of tubular type $(2, 4, 4)$,
- Self-injective algebras of tubular type $(2, 3, 6)$.

The AR-quiver of a self-injective algebra of tubular type is of the following form:

The tubular type case



The tubular type case

The following lemma indicates the relationship between the tubular type and the the rank of Grothendieck group.

Lemma 8

Let A be a non-simple, indecomposable self-injective algebra of tubular type. Then

1. *If A is of type $(2, 2, 2, 2)$, then the rank of Grothendieck group is greater than or equal to* 2.
2. *If A is of type $(3, 3, 3)$, then the rank of Grothendieck group is greater than or equal to* 2.
3. *If A is of type $(2, 4, 4)$, then the rank of Grothendieck group is greater than or equal to* 3.
4. *If A is of type $(2, 3, 6)$, then the rank of Grothendieck group is greater than or equal to* 5.

The tubular type case

The following lemma indicates relationship between the number of simple modules and the rank of quasi-tubes.

Lemma 9 (Key lemma)

Let A be a non-simple self-injective algebra with n simple modules and let $\mathcal{T}$ be in $\mathcal{Q}_p$. We have the following:

- (a) *If $n = 2$ and $p = 3$, then the number of simple modules in $\mathcal{T}$ is strictly less than 2.*
- (b) *If $n = 3$ and $p = 4$, then the number of simple modules in $\mathcal{T}$ is strictly less than 3.*
- (c) *If $n = 5$ and $p = 6$, then the number of simple modules in $\mathcal{T}$ is strictly less than 5.*

The tubular type case

Corollary 10

Let A be a non-local self-injective algebra of tubular type. Then the set of simple A-modules is not contained in a quasi-tube.

- Note that our conclusion holds for some self-injective algebras of non-polynomial growth, for example, the algebras of generalized quaternion type.

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Thank you for your attention!