Auslander-type Conditions and Weakly Gorenstein Algebras

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Contents

Introduction

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- R: an associative ring with identity
- $\operatorname{Mod} R$: the category of left R-modules
- $\operatorname{mod} R$: the category of finitely generated left R-modules
- For a module $M \in \operatorname{Mod} R$, we use $\operatorname{id}_R M$ and $\operatorname{fd}_R M$ to denote the injective and flat dimensions of M, respectively.
- For an R-module M, we use

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- (**Bass 1963**) A commutative Noetherian ring R is Gorenstein $\Longleftrightarrow \mathrm{fd}_R E^i(R) \leq i$ for any $i \geq 0$.
- (**Auslander 1975**) For a left and right Noetherian ring R, $\mathrm{fd}_R E^i({}_R R) \leq i$ for any $i \geq 0 \Longleftrightarrow \mathrm{fd}_{R^{op}} E^i(R_R) \leq i$ for any $i \geq 0$. In this case, R is said to satisfy the **Auslander condition**.

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Gorenstein Symmetry Conjecture (GSC)

For any artin algebra R, $\mathrm{id}_R R < \infty \Longleftrightarrow \mathrm{id}_{R^{op}} R < \infty$.

- **(Auslander–Reiten 1994) GSC** holds true for an artin algebra satisfying the Auslander condition.
- Non-commutative rings satisfying the Auslander condition is a non-commutative analogue of commutative Gorenstein rings. Such rings play a crucial role in homological theory, representation theory of algebras and non-commutative algebraic geometry, and others. It is also interesting from the viewpoint of some unsolved homological conjectures, e.g. the finitistic dimension conjecture, the (generalized) Nakayama conjecture, the Gorenstein symmetry conjecture, and so on.

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Aims

(1) Introduce and study modules satisfying Auslander-type conditions, and give a conversion of **AGC**.

(2) Give some equivalent characterizations of (weakly) Gorenstein algebras in terms of Gorenstein projective modules and modules satisfying Auslander-type conditions.

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This talk is based on the following two papers:

(1) Z. Y. Huang, *On Auslander-type conditions of modules*, Publ. Res. Inst. Math. Sci. **59** (2023), 57–88.

(2) Z. Y. Huang, *Auslander-type conditions and weakly Gorenstein algebras*, Bull. London Math. Soc. (to appear)

Modules satisfying Auslander-type conditions

2. Modules satisfying Auslander-type conditions

Definition 2.1.

Let $M \in \operatorname{Mod} R$ and let $m, n \geq 0$. Then M is said to be $G_n(m)$ if $\operatorname{fd}_R E^i(M) \leq m + i$ for any $0 \leq i \leq n-1$, and M is said to be $G_\infty(m)$ if it is $G_n(m)$ for all n. In particular, M is said to satisfy the **Auslander condition** if it is $G_\infty(0)$.

Example 2.2.

Let R be a left and right Noetherian ring. Then we have

(1) ${}_RR$ is $G_n(m) \iff$ the ring R is $G_n(m)^{op}$ in the sense of H–Iyama (2007).

(2) Let $\operatorname{id}_{R^{op}} R = m (< \infty)$. Then $\operatorname{fd}_R E \leq m$ for any injective left R-module E **(Iwanaga 1980)**. So any module in $\operatorname{Mod} R$ is $G_\infty(m)$.

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Example 2.2. (continued)

(3) Let K be an algebraically closed field, and let Q be the quiver

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n+1$$

and $R = KQ/J^2$, where J is the Jacobson radical of KQ. Then $\mathrm{gl.dim}\, R = n$. Moreover, we have that $S(1)$ is $G_{n+1}(0)$ and hence $G_\infty(0)$, and $S(i)$ is both $G_{n-i+1}(0)$ and $G_\infty(i-1)$ for any $2 \le i \le n+1$.

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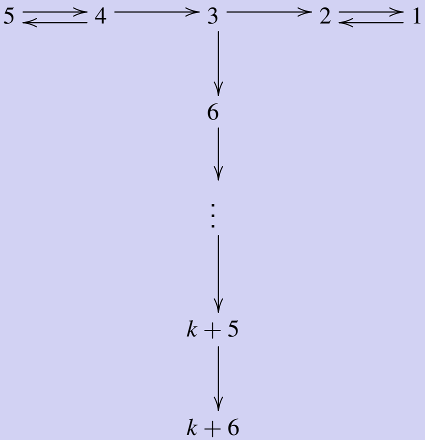

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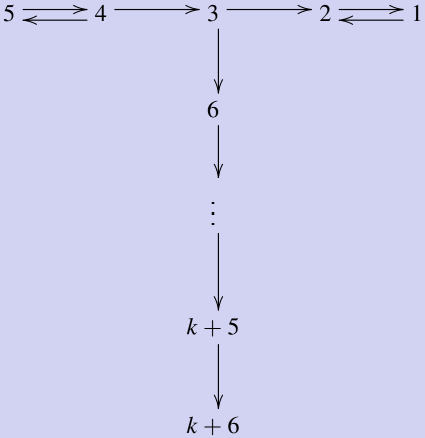

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(i) none of ${}_RR$, $P(1)$, $P(3)$ and $P(4)$ is $G_n(m)$ for any $n, m \geq 0$;

(ii) both $P(2)$ and $P(5)$ are injective;

(iii) for any $7 \leq i \leq k+6$, $S(i)$ is $G_{i-6}(0)$ but not $G_{i-5}(0)$, and $S(6)$ is not $G_n(m)$ for any $n, m \geq 0$.

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- For any $m \geq 0$, we write

$$\mathcal{G}_\infty(m) := \{M \in \operatorname{Mod} R \mid M \text{ is } G_\infty(m)\}.$$

- We have the following inclusion chain

$$\mathcal{G}_\infty(0) \subseteq \mathcal{G}_\infty(1) \subseteq \cdots \subseteq \mathcal{G}_\infty(m) \subseteq \cdots.$$

- For an R-module M, we use

$$\cdots \to F_i(M) \xrightarrow{\pi_i(M)} \cdots \xrightarrow{\pi_2(M)} F_1(M) \xrightarrow{\pi_1(M)} F_0(M) \xrightarrow{\pi_0(M)} M \to 0$$

to denote a minimal flat resolution of M, where $\pi_i(M) : F_i(M) \to \operatorname{Im} \pi_i(M)$ is a flat cover of $\operatorname{Im} \pi_i(M)$ for any $i \geq 0$.

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Theorem 2.3.

For a left Noetherian ring R, consider the following conditions.

(1) ${}_RR$ satisfies the Auslander condition (that is, ${}_RR \in \mathcal{G}_\infty(0)$).

(2) Any flat left R-module satisfies the Auslander condition.

(3) $\mathrm{fd}_R\, E^i(M) \leq \mathrm{fd}_R\, M + i$ for any left R-module M and $i \geq 0$.

(4) $\mathrm{fd}_R\, E^0(M) \leq \mathrm{fd}_R\, M$ for any left R-module M.

(5) $\mathrm{id}_{R^{op}}\, F_i(E) \leq i$ for any injective right R-module E and $i \geq 0$.

(6) $\mathrm{id}_{R^{op}}\, F_i(N) \leq \mathrm{id}_{R^{op}}\, N + i$ for any right R-module N and $i \geq 0$.

(7) $\mathrm{id}_{R^{op}}\, F_0(N) \leq \mathrm{id}_{R^{op}}\, N$ for any right R-module N.

We have $(1) \Longleftrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longrightarrow (5) \Longleftrightarrow (6) \Longleftrightarrow (7)$. If R is further right Noetherian, then all of the above and below conditions are equivalent.

$(i)^{op}$ The opposite version of (i) $(1 \leq i \leq 7)$.

When R is a right coherent and left Noetherian projective K-algebra over a commutative ring K, the equivalence $(1) \Longleftrightarrow (4)$ has been known **(Miyachi 2000)**.

Theorem 2.3.

For a left Noetherian ring R, consider the following conditions.

(1) ${}_RR$ satisfies the Auslander condition (that is, ${}_RR \in \mathcal{G}_\infty(0)$).

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The following result is a conversion of **AGC**.

Theorem 2.4

Let R be an artin algebra satisfying the Auslander condition. TFAE.

(1) R is Gorenstein.

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(Weakly) Gorenstein algebras

3. (Weakly) Gorenstein algebras

In this section, R is an artin algebra.

Recall that a module $M \in \operatorname{mod} R$ is called **Gorenstein projective** if there exists an exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

in $\operatorname{mod} R$ with all P_i, P^i projective, such that it remains exact after applying the functor $\operatorname{Hom}_R(-, R)$ and $M \cong \operatorname{Im}(P_0 \to P^0)$.

- We write

$$\mathcal{GP}(\operatorname{mod} R) := \{\text{Gorenstein projective modules in } \operatorname{mod} R\}.$$

- For any $s \geq 0$, we write

$$\mathcal{GP}(\operatorname{mod} R)^{\leq s} := \{M \in \operatorname{mod} R \mid \text{G-pd}_R M \leq s\}.$$

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Theorem 3.1.

TFAE for any $m \geq 0$.

(1) ${}_RR \in \mathcal{G}_\infty(m)$ and R is Gorenstein.

(2) ${}_RR \in \mathcal{G}_\infty(m)$ and $\mathrm{id}_R R < \infty$.

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$$^{\perp}{}_{R}R := \{M \in \operatorname{mod} R \mid \operatorname{Ext}_R^{\geq 1}(M, R) = 0\}.$$

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Let M be an R-module. An injective coresolution

$$0 \to M \to E^0 \xrightarrow{\delta^1} E^1 \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^n} E^n \xrightarrow{\delta^{n+1}} \cdots$$

is called **ultimately closed** if there exists some n such that $\operatorname{Im}\delta^n = \oplus W_j$ with each W_j isomorphic to a direct summand of some $\operatorname{Im}\delta_{i_j}$ with $i_j < n$. An artin algebra R is said to be of **ultimately closed type** if the minimal injective coresolution of any finitely generated left R-module is ultimately closed **(Tachikawa 1973)**.

Remark 3.2.

(1) The following classes of algebras are left weakly Gorenstein.

- **(1.1)** algebras R such that ${}_RR$ admits an ultimately closed injective coresolution (in particular, R is of ultimately closed type);
- **(1.2)** **(Beligiannis 2011)** algebras R such that ${}^{\perp}{}_RR$ is of finite type.

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Remark 3.2. (continued)

(2) **(Tachikawa 1973; Ringel 2012; Ringel–Zhang 2020)** The class of algebras of ultimately closed type includes

- (2.1) algebras with radical square zero;
- (2.2) representation-finite algebras;
- (2.3) algebras R with Loewy length m such that R/J^{m-1} is representation-finite, where J is the Jacobson radical of R.
- (2.4) the class of torsionless-finite algebras which includes:
  - (a) algebras R with $R/\operatorname{Soc}(R_R)$ representation-finite, where $\operatorname{Soc}(R_R)$ is the socle of R_R;
  - (b) Minimal representation-infinite algebras;
  - (c) algebras stably equivalent to hereditary algebras;
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The equivalence $(1) \Longleftrightarrow (3)$ in this result has been known **(Wang–Li–Wu–Hu 2023).**

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Marczinzik (2016) posed the following question: Is a left weakly Gorenstein artin algebra also right weakly Gorenstein? For the sake of convenience, we state this question as the following conjecture.

Weakly-Gorenstein Symmetry Conjecture (WGSC)

An artin algebra is left weakly Gorenstein $\Longleftrightarrow$ it is right weakly Gorenstein.

Observation. (Ringel–Zhang 2020)

WGSC $\Longrightarrow$ GSC.

Proof. Suppose that **WGSC** holds. Let $\mathrm{id}_R R = n < \infty$. Then R is right weakly Gorenstein, and hence is left weakly Gorenstein. It follows that any n-syzygy module in $\mathrm{mod}\, R$ is in ${}^{\perp}{}_R R = \mathcal{GP}(\mathrm{mod}\, R)$. So G-pd$_R M \leq n$ for any $M \in \mathrm{mod}\, R$, and hence R is n-Gorenstein **(Enochs–Jenda 2000)**. Symmetrically, if $\mathrm{id}_{R^{op}} R = n$, then R is n-Gorenstein.

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Conclusions.

(1) ${}_RR \in \mathcal{G}_\infty(m)$ and $R_R \in \mathcal{G}_\infty(m')^{op}$ with $m, m' \geq 0$: both **WGSC** and **GSC** hold (Theorem 3.3).

(2) ${}_RR \in \mathcal{G}_\infty(1)$ (that is, R is quasi Auslander): **GSC** holds, but we do not know whether **WGSC** holds or not (Theorem 3.4).

(3) ${}_RR \in \mathcal{G}_\infty(0)$ (that is, R satisfies the Auslander condition): both **WGSC** and **GSC** hold (Corollary 3.5).
Note that **GSC** holds true for an artin algebra R satisfying the Auslander condition has been proved by Auslander–Reiten (1994). Moreover, we have that R is Gorenstein $\Longleftrightarrow$ it is left or right weakly Gorenstein. This is a reduction of **AGC**, since Gorenstein algebras are left and right weakly Gorenstein, but the converse does not hold true in general (Remark 3.2).

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(1) ${}_RR \in \mathcal{G}_\infty(m)$ and $R_R \in \mathcal{G}_\infty(m')^{op}$ with $m, m' \geq 0$: both **WGSC** and **GSC** hold (Theorem 3.3).

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Thank you!