Periodic derived Hall algebras of hereditary abelian categories

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- For example, $D^b(\mathcal{A})$ is left homologically finite.

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However, the derived Hall algebras of **odd** periodic triangulated categories have been defined by Xu-Chen.

[Xu-Chen] F. Xu, X. Chen, Hall algebras of odd periodic triangulated categories, Algebr. Represent. Theory **16**(3) (2013), 673–687.

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is defined by $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ for any $\alpha, \beta \in K(\mathcal{A})$.

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In what follows, we fix a positive integer m, and denote the quotient ring $\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$.

Relations between the extensions in $D_m(\mathcal{A})$ and $D^b(\mathcal{A})$

For any triangle in $D_m(\mathcal{A})$

$$\bigoplus_{i\in\mathbb{Z}_m} \widetilde{B}_i[i] \longrightarrow \bigoplus_{i\in\mathbb{Z}_m} \widetilde{M}_i[i] \longrightarrow \bigoplus_{i\in\mathbb{Z}_m} \widetilde{A}_i[i] \longrightarrow \bigoplus_{i\in\mathbb{Z}_m} \widetilde{B}_i[i+1]$$

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such that $A_i, B_i, M_i \in \mathcal{A}$ with $i \in \mathbb{Z}_m$, by considering the homologies, we have the following periodic exact sequence in $\mathcal{A}$

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$$B_i \oplus I_{i-1}[-1] \longrightarrow M_i \longrightarrow I_i[1] \oplus A_i.$$

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Proposition [Z. 2023]

For any $A_i, B_i, M_i \in \mathcal{A}$ with $i \in \mathbb{Z}_m$, we have that

$$|\mathrm{Ext}\,^1_{D_m(\mathcal{A})}(\bigoplus_{i\in\mathbb{Z}_m} A_i[i], \bigoplus_{i\in\mathbb{Z}_m} B_i[i])_{\bigoplus_{i\in\mathbb{Z}_m} M_i[i]}|$$

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where for an object $X \in \mathcal{A}$ we set $a_X = |\mathrm{Aut}_{\mathcal{A}}(X)|$.

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$$= \sum_{[I_0],[I_1],\ldots,[I_{m-1}]\in\mathrm{Iso}(\mathcal{A})} \prod_{i\in\mathbb{Z}_m} \frac{|\mathrm{Ext}^1_{D^b(\mathcal{A})}(I_i[1]\oplus A_i, B_i \oplus I_{i-1}[-1])_{M_i}|}{a_{I_i}},$$

where for an object $X \in \mathcal{A}$ we set $a_X = |\mathrm{Aut}_{\mathcal{A}}(X)|$.

Hence, we can use the (dual) derived Hall numbers $H^{M_i}_{I_i[1]\oplus A_i, B_i\oplus I_{i-1}[-1]}$ with $i \in \mathbb{Z}_m$ to define the Hall numbers of m-periodic derived Hall algebras.

The m-periodic extended derived Hall algebra

Definition [Z. 2023]

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The Hall algebra $\mathcal{DH}^{\mathrm{e}}_m(\mathcal{A})$, called the *m-periodic extended derived Hall algebra* of $\mathcal{A}$, is the $\mathbb{C}$-vector space with the basis $\{u_{\bigoplus_{i\in\mathbb{Z}_m} M_i[i]} \prod_{i\in\mathbb{Z}_m} K_{\alpha_i,i} \mid [M_i] \in \mathrm{Iso}(\mathcal{A}), \alpha_i \in K(\mathcal{A}) \text{ for all } i \in \mathbb{Z}_m\}$,

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$$(u_{\bigoplus_{i\in\mathbb{Z}_m} A_i[i]} \prod_{i\in\mathbb{Z}_m} K_{\alpha_i,i})(u_{\bigoplus_{i\in\mathbb{Z}_m} B_i[i]} \prod_{i\in\mathbb{Z}_m} K_{\beta_i,i}) =$$

$$v^{a_0} \sum_{[I_i],[M_i]\in\mathrm{Iso}(\mathcal{A}),i\in\mathbb{Z}_m} v^{-(\hat{I}_{m-1},\alpha_0+\beta_0)+\sum_{i=1}^{m-1}(\hat{I}_i,\alpha_{i-1}+\beta_{i-1})+\sum_{i\in\mathbb{Z}_m}\langle \hat{M}_i-\hat{M}_{i+1},\hat{I}_i\rangle+\sum_{i=1}^{m-1}\langle\hat{I}_{i-1},\hat{I}_i\rangle-\langle\hat{I}_0,\hat{I}_{m-1}\rangle}$$

$$\prod_{i\in\mathbb{Z}_m} \frac{H^{M_i}_{I_i[1]\oplus A_i, B_i\oplus I_{i-1}[-1]}}{a_{I_i}} u_{\bigoplus_{i\in\mathbb{Z}_m} M_i[i]} \prod_{i\in\mathbb{Z}_m} K_{\hat{I}_i+\alpha_i+\beta_i,i},$$

where $a_0 = \sum_{i\in\mathbb{Z}_m}\langle\hat{A}_i,\hat{B}_i\rangle + \sum_{i\in\mathbb{Z}_m}(\alpha_i,\hat{B}_i - \hat{B}_{i+1}) + \sum_{i=1}^{m-1}(\alpha_i,\beta_{i-1}) - (\alpha_{m-1},\beta_0)$.

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In particular, we have that

$$u_{\bigoplus_{i\in\mathbb{Z}_m} A_i[i]} u_{\bigoplus_{i\in\mathbb{Z}_m} B_i[i]} = v^{\sum_{i\in\mathbb{Z}_m}\langle \hat{A}_i, \hat{B}_i\rangle} \sum_{[I_i],[M_i]\in \mathrm{Iso}(\mathcal{A}), i\in\mathbb{Z}_m} v^{\sum_{i\in\mathbb{Z}_m}\langle \hat{M}_i - \hat{M}_{i+1}, \hat{I}_i\rangle + \sum_{i=1}^{m-1}\langle \hat{I}_{i-1}, \hat{I}_i\rangle - \langle \hat{I}_0, \hat{I}_{m-1}\rangle}$$

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Proposition [Z. 2023]

The m-periodic extended derived Hall algebra $\mathcal{DH}^{\mathrm{e}}_m(\mathcal{A})$ has a basis given by

$$\{\prod_{i\in\mathbb{Z}_m} u_{A_i[i]} \prod_{i\in\mathbb{Z}_m} K_{\alpha_i,i} \mid [A_i] \in \mathrm{Iso}(\mathcal{A}), \alpha_i \in K(\mathcal{A}) \text{ for all } i \in \mathbb{Z}_m\}.$$

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The m-periodic extended derived Hall algebra $\mathcal{DH}^{\mathrm{e}}_m(\mathcal{A})$ is isomorphic to Bridgeland's Hall algebra (if $\mathcal{A}$ has enough projectives) and semi-derived Ringel-Hall algebras of m-periodic complexes.

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Theorem [Z. 2023]

The 2-periodic extended derived Hall algebra $\mathcal{DH}_2^{\mathrm{e}}(\mathcal{A})$ is isomorphic to the Drinfeld double Hall algebra of $\mathcal{A}$.

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$$= v^{\sum_{i\in\mathbb{Z}_m}\langle \sum_{k=0}^{m-1}(-1)^k \hat{A}_{i+k},\hat{B}_i\rangle} \sum_{[I_i],[M_i]\in\mathrm{Iso}(\mathcal{A}),i\in\mathbb{Z}_m} \prod_{i\in\mathbb{Z}_m} \frac{H^{M_i}_{I_i[1]\oplus A_i,B_i\oplus I_{i-1}[-1]}}{a_{I_i}} u_{\bigoplus_{i\in\mathbb{Z}_m} M_i[i]}.$$

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If $\mathcal{A}$ is a hereditary abelian category with Euler form **skew symmetric**, i.e. $(\alpha, \beta) = 0$ for any $\alpha, \beta \in K(\mathcal{A})$, then the K-elements are **central** in the m-periodic extended derived Hall algebra $\mathcal{DH}_m^{\mathrm{e}}(\mathcal{A})$.

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We remark that Chen, Lu and Ruan have defined the Hall algebra of root categories **without** appending K-elements.

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In fact, their Hall numbers are defined in root categories by splitting the set $\mathrm{Hom}(M, L)_{Z[1]}$ into smaller subsets $\mathrm{Hom}(M, L)_{Z[1],\delta}$ for $\delta \in K(\mathcal{A})$.

Odd periodic derived Hall algebras

Theorem [Z. 2023]

Let m be an odd positive integer. The m-periodic derived Hall algebra $\mathcal{DH}_m(\mathcal{A})$ is an associative algebra and isomorphic to the derived Hall algebra of $D_m(\mathcal{A})$ defined by Xu-Chen.

Odd periodic derived Hall algebras

Corollary [Z. 2023]

Let m be an odd positive integer. For any $A_i, B_i, C_i, M_i \in \mathcal{A}$ with $i \in \mathbb{Z}_m$, we have that

$$\sum_{[I_i],[X_i],[J_i]\in \mathrm{Iso}(\mathcal{A}),i\in\mathbb{Z}_m} q^{-\sum_{i\in\mathbb{Z}_m}\langle \hat{I}_{i-1},\hat{C}_i\rangle} \prod_{i\in\mathbb{Z}_m}\Big(\frac{H^{X_i}_{I_i[1]\oplus A_i,B_i\oplus I_{i-1}[-1]}}{a_{I_i}}\frac{H^{M_i}_{J_i[1]\oplus X_i,C_i\oplus J_{i-1}[-1]}}{a_{J_i}}\Big)$$

$$=\sum_{[I_i],[Y_i],[J_i]\in \mathrm{Iso}(\mathcal{A}),i\in\mathbb{Z}_m} q^{-\sum_{i\in\mathbb{Z}_m}\langle \hat{A}_i,\hat{J}_i\rangle} \prod_{i\in\mathbb{Z}_m}\Big(\frac{H^{M_i}_{I_i[1]\oplus A_i,Y_i\oplus I_{i-1}[-1]}}{a_{I_i}}\frac{H^{Y_i}_{J_i[1]\oplus B_i,C_i\oplus J_{i-1}[-1]}}{a_{J_i}}\Big)$$

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This formula may be viewed as the odd periodic version of Green's formula.

An algebra embedding

Let m be an odd positive integer.

Theorem [Z.-Zhang-Zhu 2023]

There exists an embedding of algebras $\varphi_m : \mathcal{DH}_m(\mathcal{A}) \to \mathcal{DH}^e_m(\mathcal{A})$ defined by

$$u_{M_0} \mapsto u_{M_0} K_{-\frac{1}{2}\hat{M}_0}, \text{ if } m = 1;$$

and

$$u_{\bigoplus_{i\in\mathbb{Z}_m} M_i[i]} \mapsto v^{\frac{1}{4}\sum_{i=1}^{m-1}\langle \sum_{k=0}^{m-1}(-1)^k\hat{M}_{i+k}, \sum_{k=0}^{m-1}(-1)^k\hat{M}_{i+1+k}\rangle - \frac{1}{4}\langle \sum_{k=0}^{m-1}(-1)^k\hat{M}_{1+k}, \sum_{k=0}^{m-1}(-1)^k\hat{M}_{k}\rangle + \sum_{i=0}^{m-1}\langle \hat{M}_i, \sum_{k=1}^{m-1}(-1)^k\hat{M}_{i+k}\rangle}$$

$$u_{\bigoplus_{i\in\mathbb{Z}_m} M_i[i]} \prod_{i\in\mathbb{Z}_m} K_{-\frac{1}{2}\sum_{k=0}^{m-1}(-1)^k\hat{M}_{i+1+k}, i}, \text{ if } m > 1;$$

for any $M_i \in \mathcal{A}$ with $i \in \mathbb{Z}_m$.

Revisit: *m*-periodic extended derived Hall algebra

Definition

The Hall algebra $\mathcal{DH}^{\mathrm{e}}_m(\mathcal{A})$, called the *m-periodic extended derived Hall algebra* of $\mathcal{A}$, is the $\mathbb{C}$-vector space with the basis $\{u_{\bigoplus_{i\in\mathbb{Z}_m} M_i[i]} \prod_{i\in\mathbb{Z}_m} K_{\alpha_i,i} \mid [M_i]\in \mathrm{Iso}(\mathcal{A}), \alpha_i \in \frac{1}{2}K(\mathcal{A}) \text{ for all } i\in\mathbb{Z}_m\}$, and with the multiplication defined on basis elements by

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Compared with [LP], our homomorphism is defined on the basis elements of $\mathcal{DH}_m(\mathcal{A})$, rather than just on generating elements.

[LP] J. Lin, L. Peng, Semi-derived Ringel-Hall algebras and Hall algebras of odd-periodic relative derived categories, Scinence China Mathematics, online.

Thank you!