# Inhomogeneous tubes and a conjecture by Geiss-Leclerc-Schröer on root systems

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# §1 Backgrounds

**Gabriel's Theorem**

If $Q$ is a connected quiver, then there are only finitely many isomorphism classes of indecomposable representations if and only if $Q$ is a Dynkin quiver of type $A_n, D_n, E_6, E_7, E_8$. In this case, the assignment $X \mapsto \underline{\dim} X$ induces a bijection between the isomorphism classes of indecomposable representations of $Q$ and the positive roots of the corresponding simple complex Lie algebra.

Generalizations of Gabriel's Theorem (simply laced):

(1) 1973, Nazarova, Donovan-Freislich: $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$

(2) 1980, Kac: general quivers

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Let $Q$ be a quiver. Then there is a bijection between the set of dimension vectors of indecomposable representations of $Q$ and the set of positive roots of the corresponding Kac-Moody Lie algebra.

Generalizations of Gabriel's Theorem (simply laced and non-simply laced):

(1) 1976, Dlab-Ringel: valued quivers.

(2) 2017, Geiss-Leclerc-Schröer: a class of Iwanaga-Gorenstein algebras.

Given a symmetrizable generalized Cartan matrix $C$ with a symmetrizer $D$ and an acyclic orientation $\Omega$ of $C$, Geiss-Leclerc-Schröer introduced a quiver $Q = Q(C, \Omega)$ and a finite-dimensional $K$-algebra $H = H(C, D, \Omega) = KQ/I$, where $K$ is a field and the ideal $I$ is generated by some powers of loops and some commutative relations.

## Example

Let $C = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$, $D = diag(2,1,2), \Omega = \{(2,1),(3,2)\}$.

Then (1) $H = H(C, D, \Omega) = KQ/I$, where

$$Q : \varepsilon_1 \circlearrowleft 1 \xrightarrow{\alpha_{21}} 2 \xrightarrow{\alpha_{32}} 3 \circlearrowleft \varepsilon_3$$

and $I =< \varepsilon_1^2, \varepsilon_3^2 >$.

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(2) $H = H(C, 2D, \Omega) = KQ/I$, where

$$Q : \varepsilon_1 \circlearrowleft 1 \xrightarrow{\alpha_{21}} \varepsilon_2 \circlearrowleft 2 \xrightarrow{\alpha_{32}} 3 \circlearrowleft \varepsilon_3$$

and $I = < \varepsilon_1^4, \varepsilon_2^2, \varepsilon_3^4, \alpha_{21}\varepsilon_1^2 - \varepsilon_2\alpha_{21}, \alpha_{32}\varepsilon_2 - \varepsilon_3^2\alpha_{32} >$.

Let $e_i$ be the idempotent in $H$ corresponding to the vertex $i$ in $Q$ and $H_i = e_i H e_i$. In the above example (1),

$$H_i \cong \begin{cases} K[t]/\langle t^2 \rangle, & i = 1, 3 \\ K, & i = 2 \end{cases}$$

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**Definition**

(1) A left $H$-module $M$ is called *locally free* if each $M_i = e_i M$ is a free $H_i$-module for all $i$. Denote by $r_i$ the rank of the free $H_i$-module $M_i$. Then $\underline{\text{rank}}(M) = (r_1, \cdots, r_n)$ is called the *rank vector* of $M$.

(2) An indecomposable $H$-module $M$ is called *$\tau$-locally free* if $\tau^k(M)$ is locally free for all $k \in \mathbb{Z}$.

## Geiss-Leclerc-Schröer's Theorem

There are only finitely many isoclasses of $\tau$-locally free $H$-modules if and only if $C$ is of Dynkin type. In this case, the assignment $M \mapsto \underline{\text{rank}}(M)$ provides a bijection between the set of isomorphism classes of $\tau$-locally free $H$-modules and the set of positive roots of the corresponding simple complex Lie algebras.

**Remark**

*When $C$ is of Dynkin type, Geiss-Leclerc-Schröer applied locally-free $H$-modules to construct*

*(1) the enveloping algebra of the positive part of a semisimple complex Lie algebra.*

*(2) cluster variables of cluster algebras of finite type.*

## Geiss-Leclerc-Schröer's Conjecture

There is a bijection between the set of positive roots of the Kac-Moody Lie algebra $g(C)$ associated with $C$ and the set of rank vectors of $\tau$-locally free $H$-modules.

### Geiss-Leclerc-Schröer's Conjecture

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### Remark

*(1) If $C$ is symmetric and $D$ is the identity matrix, then the conjecture is true by Kac's Theorem.*

*(2) If $C$ is of Dynkin type and $D$ is arbitrary, then the conjecture is true by Geiss-Leclerc-Schröer's Theorem.*

*(3) Geiss-Leclerc-Schröer proved that there is a bijection between the isomorphism classes of rigid $\tau$-locally free $H$-modules and the set of positive real schur roots of $(C, \Omega)$.*

# §2 Representation theory on affine type algebras $H = H(C, D, \Omega)$

§2.1 Type $\widetilde{C}_n$

Assume that the Cartan matrix $C$ is of type $\widetilde{C}_n$, that is,

$$C = \begin{pmatrix} 2 & -1 & & & & & \\ -2 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -2 \\ & & & & & -1 & 2 \end{pmatrix}_{(n+1)\times(n+1)}.$$

Note that $D = \mathrm{diag}(2, 1, 1, \cdots, 1, 1, 2)$ is a symmetrizer of $C$.

For any orientation $\Omega$ of $C$, the algebra $H = H(C, D, \Omega)$ is a string algebra. Moreover, $H$ is a gentle algebra. These algebras are called *string algebras of type $\widetilde{C}_n$*.

For example, assume that $\Omega$ is a linear orientation of $C$, then the algebra $H = KQ/I$, where

$$Q: \ \varepsilon_1 \circlearrowleft 1 \xrightarrow{\alpha_{21}} 2 \xrightarrow{\alpha_{32}} \cdots \xrightarrow{\alpha_{n+1,n}} n+1 \circlearrowleft \varepsilon_{n+1}$$

and $I = < \varepsilon_1^2, \varepsilon_{n+1}^2 >$.

## Theorem [Bulter-Ringel]

Let $A$ be a string algebra. Then the following hold.

(1) Any indecomposable $A$-module is either a string or band module.

(2) The number of middle terms in an Auslander-Reiten sequence is either one or two.

## Theorem [Huang-Lin-Su]

Let $H$ be a string algebra of type $\widetilde{C}_n$. Then the Auslander-Reiten quiver $\Gamma_H$ of $H$ consists of the following:

(1) one component $\mathcal{T}_{PI}$ containing all the indecomposable preprojective modules and all the indecomposable preinjective modules;

(2) one tube of rank $n$;

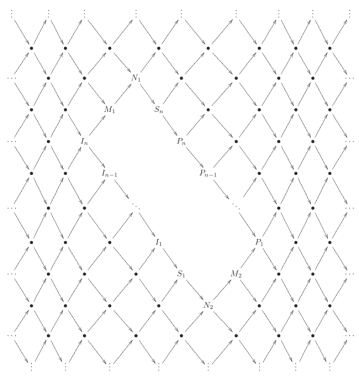
(3) homogeneous tubes $\mathcal{H}_{w,S}$, where $w$ runs through a complete set of representatives of bands, $S$ runs through all isoclasses of simple modules over $K[T, T^{-1}]$;

(4) infinitely many components of type $\mathbb{Z}A_{\infty}^{\infty}$, which do not contain $\tau$-locally free modules.

# Eaxmple

$$Q: \ \varepsilon_1 \circlearrowleft 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} n \circlearrowleft \varepsilon_n$$

The component $\mathcal{T}_{PI}$ is as follows.



## Theorem [Huang-Lin-Su]

Let $H$ be a string algebra of type $\widetilde{C}_n$ and $M$ be an indecomposable $H$-module. Then $M$ is $\tau$-locally free if and only if one of the following is satisfied:

(1) $M$ is preprojective.

(2) $M$ is preinjective.

(3) $M$ is a regular module occurring in any tube.

**Theorem [Huang-Lin-Su]**

Let $H = H(C, D, \Omega)$ be a string algebra of type $\widetilde{C}_n$.

(1) If $M$ is a $\tau$-locally free $H$-module, then $\underline{\text{rank}}(M)$ is a positive root of $C$.

(2) If $\alpha$ is a positive real root of $C$, there is a unique $\tau$-locally free $H$-module $M$ with $\underline{\text{rank}}(M) = \alpha$.

(3) If $\alpha$ is a positive imaginary root of $C$, there are infinitely many $\tau$-locally free $H$-modules $M$ with $\underline{\text{rank}}(M) = \alpha$.

### Theorem [Huang-Lin-Su]

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### Corollary [Huang-Lin-Su]

Let $C$ be of type $\widetilde{C}_n$ and $D = \text{diag}(2, 1, 1, \cdots, 1, 1, 2)$. Then GLS's Conjecture is true.

# §2.2 Affine types: the general case

**Proposition [Dlab-Ringel]**

Let $\mathbf{i} = (i_1, \ldots, i_n)$ be a $+$-admissible sequence with respect to $(C, \Omega)$. The set of positive roots of type $C$ is the disjoint union of preprojective, preinjective and regular roots as follows.

(1) $\{c_{\mathbf{i}}^{-r}(\beta_{\mathbf{i},k}) \mid r \in \mathbb{Z}_{\geq 0}, 1 \leq k \leq n\}$.

(2) $\{c_{\mathbf{i}}^{s}(\gamma_{\mathbf{i},k}) \mid s \in \mathbb{Z}_{\geq 0}, 1 \leq k \leq n\}$.

(3) $\{x_0 + rg\delta \mid x_0 \text{ is a positive root } \leq g\delta,\ r \in \mathbb{Z}_{\geq 0}\}$, where $1 \leq g \leq 3$ is a constant and $x_0$ can be deduced from [Table 6, Dlab-Ringel].

**Proposition [Lin-Su]**

Let $M$ be a $\tau$-locally free regular $H$-module. Then there exists some positive integer $N$ such that $c^N(\underline{\mathrm{rank}}\, M) = \underline{\mathrm{rank}}\, M$.

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**Proposition [Lin-Su]**

Let $\mathcal{C}$ be a connected component of $\Gamma_H$ that contains a regular $\tau$-locally free $H$-module. Then $\mathcal{C}$ is either a tube or of the form $\mathbb{Z}\mathbb{A}_\infty$. Furthermore, if $\mathcal{C}$ contains a $\tau$-periodic module, then $\mathcal{C}$ is a tube.

### Theorem [Lin-Su]

Let $C$ be a generalized Cartan matrix of affine type and $H = H(C, D, \Omega)$. Then for any positive root $\alpha$ of the Kac-Moody Lie algebra $g(C)$, there exists a $\tau$-locally free left $H$-module $M$ such that $\underline{\text{rank}}(M) = \alpha$.

### Theorem [Lin-Su]

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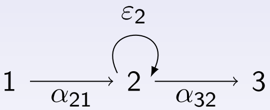
Idea of the proof: (1) It is independent on the orientation $\Omega$.

(2) It is independent on the symmetriser $D$.

(3) *For each inhomogeneous tube $\mathcal{C}$ in [Section 6, Dlab-Ringel], there is a good tube of $\tau$-locally free $H$-modules of the same rank such that the rank vectors of the mouth modules are exactly the same as the dimension vectors of the mouth modules of $\mathcal{C}$.*

Let $C = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$, $D = \mathrm{diag}(1,2,1)$ and $\Omega = \{(2,1),(3,2)\}$. Thus $C$ is a Cartan matrix of affine type $\widetilde{B}_2$ with a minimal symmetrizer. Then $H = H(C, D, \Omega)$ is given by the quiver




with relation $\varepsilon_2^2 = 0$.

Consider the following locally free $H$-module

$$N = \begin{array}{ccccc} & & 2 & & \\ & & \downarrow & & \\ 1 & \longrightarrow & 2 & \longrightarrow & 3. \end{array}$$

**Proposition**

*$N$ is a $\tau$-locally free $H$-module at the bottom of a stable tube of rank 2, and $\underline{\text{rank}}(N)$ is a minimal positive imaginary root $\delta$.*

In fact,

$$\tau N = \begin{array}{ccccc} 1 & \longrightarrow & 2 & \longrightarrow & 3. \\ & & \downarrow & & \\ & & 2 & & \end{array}$$

Consider the following locally free $H$-module

$$
M = \quad \begin{array}{ccccc}
 & & 2 & & \\
 & & \downarrow & & \\
1 & \longrightarrow & 2 & \longrightarrow & 3. \\
 & \searrow & & & \\
 & & 2 & & \\
 & & \downarrow & & \\
 & & 2 & &
\end{array}
$$

There is a short exact sequence $0 \to E_2 \to M \to N \to 0$, where $E_2$ is the generalized simple $H$-module at vertex 2.

**Proposition**

*$M$ is a $\tau$-locally free $H$-module at the bottom of a stable tube of rank 2, but $\underline{\text{rank}}(M) = \delta + \alpha_2$ is not a positive root.*

## Theorem [Lin-Su]

Let $C$ be a Cartan matrix of type $\widetilde{\mathbb{B}}_n$, $\widetilde{\mathbb{CD}}_n$, $\widetilde{\mathbb{F}}_{41}$ or $\widetilde{\mathbb{G}}_{21}$ and $D$ a minimal symmetriser.

(1) The AR-quiver $\Gamma_H$ has an inhomogeneous tube of $\tau$-locally free modules, whose mouth modules are not rigid and have $\delta$ as their rank vectors.

(2) There exist $\tau$-locally free $H$-modules such that their rank vectors are not roots. Consequently, GLS's Conjecture fails in these four types.

*Thank you!*