The parity of Lusztig's restriction functor and Green's formula for a quiver with automorphism

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Hereditary Algebra

For algebra B over field k, we call it a hereditary algebra if $gl.dim(B) <= 1$

- Path algebra kQ of a quiver Q is a hereditary algebra.

Let $\mathbb{F}_q$ be a finite field of order q. Let B be a finite-dimensional hereditary algebra over $\mathbb{F}_q$.

- Λ be the set of isomorphism classes of finite-dimensional B-modules.
- For any $\alpha \in \Lambda$, we fix a B-module M_α such that $M_\alpha \in \alpha$.

The Ringle-Hall algebra $\mathcal{H}_q(modB)$ is a $\overline{\mathbb{Q}_l}$-algebra with a basis $\{u_{M_\alpha}|\alpha \in \Lambda\}$.
For algebra B, $\alpha, \beta \in \Lambda$, there is Euler form as follows.

$$\langle \alpha, \beta \rangle = \Sigma_{i \in \mathbb{N}} (-1)^i \dim \operatorname{Ext}^i(M_\alpha, M_\beta)$$

we note $(\alpha, \beta) := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$.

$v_q \in \overline{\mathbb{Q}_l}$ is a fixed square root of q.
Multiplication:

$$u_{M_\alpha} * u_{M_\beta} = \Sigma_{\gamma \in \Lambda} v_q^{\langle \alpha, \beta \rangle} g_{\alpha\beta}^{\gamma} u_{M_\gamma}$$

where

$$g_{\alpha\beta}^{\gamma} = \frac{|\mathrm{Ext}_{\mathrm{B}}^{1}(\mathrm{M}_\alpha, \mathrm{M}_\beta)_{\mathrm{M}_\gamma}|}{|\mathrm{Hom}_{\mathrm{B}}(\mathrm{M}_\alpha, \mathrm{M}_\beta)|} \frac{|\mathrm{Aut}(\mathrm{M}_\gamma)|}{|\mathrm{Aut}_{\mathrm{B}}(\mathrm{M}_\alpha)||\mathrm{Aut}_{\mathrm{B}}(\mathrm{M}_\beta)|}$$

is the filtration number. $\mathrm{Ext}_{\mathrm{B}}^{1}(\mathrm{M}_\alpha, \mathrm{M}_\beta)_{\mathrm{M}_\gamma}$ consists of extension where the middle term is isomorphic to M_γ.
$a_\alpha = |\mathrm{Aut}(\mathrm{M}_\alpha)|$.

Comultiplication on Hall Algebra

Green defined a comultiplication on $\mathcal{H}_q(modB)$ by

$$\delta(u_{M_\gamma}) = \Sigma_{\alpha,\beta\in\Lambda} v_q^{\langle\alpha,\beta\rangle} h_\gamma^{\alpha\beta} u_{M_\alpha} \otimes u_{M_\beta}$$

where $h_\gamma^{\alpha\beta} = \frac{|\mathrm{Ext}^1_\mathrm{B}(\mathrm{M}_\alpha,\mathrm{M}_\beta)_{\mathrm{M}_\gamma}|}{|\mathrm{Hom}_\mathrm{B}(\mathrm{M}_\alpha,\mathrm{M}_\beta)|}$. It gives a bialgebra structure on Ringle-Hall algebra.

Green's Formula

Green proved a non-trivial homological formula about filtration numbers which is called Green's formula: for any $\alpha, \beta, \alpha', \beta' \in \Lambda$, we have

$$a_\alpha a_\beta a_{\alpha'} a_{\beta'} \sum_{\gamma \in \Lambda} a_\gamma^{-1} g_{\alpha\beta}^{\gamma} g_{\alpha'\beta'}^{\gamma}$$

$$= \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Lambda} \left(\frac{|\mathrm{Ext}\,^1_{kQ}(M_{\alpha_1}, M_{\beta_2})|}{|\mathrm{Hom}_{kQ}(M_{\alpha_1}, M_{\beta_2})|} g_{\alpha_1\alpha_2}^{\alpha} g_{\beta_1\beta_2}^{\beta} g_{\alpha_1\beta_1}^{\alpha'} g_{\alpha_2\beta_2}^{\beta'} a_{\alpha_1} a_{\alpha_2} a_{\beta_1} a_{\beta_2} \right)$$

If we define the multiply $\circ$ on $\mathcal{H}_q(mod(B)) \otimes \mathcal{H}_q(mod(B))$ as follows

$$(u_{M_\alpha} \otimes u_{M_\beta}) \circ (u_{M_{\alpha'}} \otimes u_{M_{\beta'}}) := v_q^{-(\alpha', \beta)} (u_{M_\alpha} * u_{M_\beta}) \otimes (u_{M_{\alpha'}} * u_{M_{\beta'}})$$

Then we will have that $\circ\delta = \delta*$, which is the same as Green's formula.

Remark

- Ringel-Hall algebra is isomorphic to the positive part of the quantized enveloping algebra of a generalized Kac-Moody Lie algebra (in the sense of Borcherds).
- In particular, the generic form of the composition subalgebra (generated by u_i corresponding to simple representations $S_i, i \in I$) of $\mathcal{H}_q(Q)$ is isomorphic to $U_v^-(\mathfrak{g})$, where $\mathfrak{g}$ is the Kac-Moody Lie algebra defined by the generalized Cartan matrix induced by the quiver Q.

From Quivers to Hereditary Algebras

For field $\mathbb{C}$, we could obtain all of the finite dimensional hereditary algebras from path algebras of quivers.
But when the field $\mathbb{F}_q$, we need to consider a quiver with automorphism and Frobenius map with this automorphism. From now on we denoted $k := \overline{\mathbb{F}_q}$

Definition

(1) A finite quiver (I, H, s, t) consists of two finite sets I, H and two maps $s, t : H \to I$, where I is the set of vertices, H is the the set of arrows, and for any $h \in H$, the images $s(h)$ and $t(h) \in I$ are its source and target respectively. A loop of the quiver is an arrow $h \in H$ satisfying $s(h) = t(h)$.
(2) Let (I, H, s, t) be a finite quiver without loops, an admissible automorphism a of the quiver consists of two permutations $a : I \to I$ and $a : H \to H$ satisfying
(a) $a(s(h)) = s(a(h)), a(t(h)) = t(a(h))$ for any $h \in H$;
(b) $s(h), t(h) \in I$ belong to different a-orbits for any $h \in H$.

Frobenius Maps

- Let V be a k-vector space, a Frobenius map on V is a $\mathbb{F}_q$-linear isomorphism $F_V : V \to V$ satisfying
(a) for any $\lambda \in k$ and $v \in V$, we have $F_V(\lambda v) = \lambda^q F_V(v)$;
(b) for any $v \in V$, there exists $n \geqslant 1$ such that $F_V^n(v) = v$.
If there exists such a Frobenius map, then the fixed point set V^{F_V} is a $\mathbb{F}_q$-subspace such that $V = k \otimes_{\mathbb{F}_q} V^{F_V}$, and we say V has a $\mathbb{F}_q$-structure.
- Let A be an algebra over k, a Frobenius morphism on A is a Frobenius map $F_A : A \to A$ on the underlying k-vector space preserving the unit and the multiplication. If there exists such a Frobenius morphism, then the fixed point set A^{F_A} is a $\mathbb{F}_q$-subalgebra such that $A = k \otimes_{\mathbb{F}_q} A^{F_A}$, and we say A has a $\mathbb{F}_q$-structure.

Frobenius Maps

- Let A be an algebra over k with the Frobenius morphism $F_A : A \to A$ and $M \in \mathrm{mod}\,_k A$, a Frobenius morphism on M is a Frobenius map $F_M : M \to M$ on the underlying k-vector space satisfying $F_M(ma) = F_M(m)F_A(a)$ for any $m \in M$ and $a \in A$. If there exists such a Frobenius morphism, then the fixed point set $M^{F_M} \in \mathrm{mod}\,_{\mathbb{F}_q}(A^{F_A})$ such that $M = k \otimes_{\mathbb{F}_q} M^{F_M}$, and we say M is F_A-stable.
- Let A be a k-algebra with the Frobenius morphism $F_A : A \to A$, we define $\mathrm{mod}\,_k^{F_A} A$ to be the category of F_A-stable modules. More precisely, its objects are of the form (M, F_M), where $M \in \mathrm{mod}\,_k A$ is F_A-stable and $F_M : M \to M$ is the Frobenius morphism; and its morphisms $(M, F_M) \to (M', F_{M'})$ are morphisms $f : M \to M'$ of A-modules satisfying $fF_M = F_{M'}f$.

Quivers to Algebras

Theorem 3.2 [Deng, Du]

There is an equivalence of categories

$$\operatorname{mod}_k^{F_A} A \xrightarrow{\simeq} \operatorname{mod}_{\mathbb{F}_q}(A^{F_A})$$

defined by $(M, F_M) \mapsto M^{F_M}$.

Theorem 6.5 [Deng, Du]

Let B be a finite-dimensional hereditary basic algebra over $\mathbb{F}_q$, then there exists a finite quiver Q without loops and an admissible automorphism a such that $B \cong (kQ)^{\tilde{F}}$, where $\tilde{F} = a \circ F_{kQ}$

Thus we have that there are equivalences of categories

$$\operatorname{mod}_k^{\tilde{F}}(kQ) \simeq \operatorname{mod}_{\mathbb{F}_q}((kQ)^{\tilde{F}}) \simeq \operatorname{mod}_{\mathbb{F}_q} B.$$

Remark

If we want to give a description of Hall algebras in sheaves complexes categories, we need to consider Hall algebras as algebras via functions.

The Moduli Space of Quiver Q

For quiver $Q = (I, H)$ and $\nu \in \mathbb{N}I$, V is a $I-$graded $k-$vector space $\underline{\dim}(V) = \nu$.

- $E_V = E_v = \bigoplus_{h \in H} \mathrm{Hom_k}(\mathrm{V_{s(h)}}, \mathrm{V_{t(h)}})$
- $G_V = G_v = \prod_{i \in I} Gl_{v_i}$

For a Cartan matrix $A = (a_{ij})_{i,j=1,\ldots,n}$, there is quiver $Q = (I, \Omega)$, where vertex set I has n elements, and $(S_i, S_j) = a_{ij}$.

It is easy to see that if we change the orientation of the quiver, the corresponding Cartan matrix would not change.

Hall Algebras Via Functions

- $\tilde{\mathcal{H}}_{G_v^{\tilde{F}}}(E_v^{\tilde{F}})$ is the set of $G_v^{\tilde{F}}$-stable functions on $E_v^{\tilde{F}}$ where $v \in \mathbb{N}Q_0$ and $\tilde{\mathcal{H}}^{\tilde{F}}(Q) := \oplus_{v\in\mathbb{N}Q_0}\tilde{\mathcal{H}}_{G_v^{\tilde{F}}}(E_v^{\tilde{F}})$ with multiplicaiton and comultiplication as follows

Multilplication

$$E_\alpha^{\tilde{F}} \times E_\beta^{\tilde{F}} \xleftarrow{p_1} (E')_{\alpha+\beta}^{\tilde{F}} \xrightarrow{p_2} (E'')_{\alpha+\beta}^{\tilde{F}} \xrightarrow{p_3} E_{\alpha+\beta}^{\tilde{F}}$$

$$m_{\alpha,\beta}^{\tilde{F}} : \tilde{\mathcal{H}}_{G_\alpha^{\tilde{F}}}(E_\alpha^{\tilde{F}}) \times \tilde{\mathcal{H}}_{G_\beta^{\tilde{F}}}(E_\beta^{\tilde{F}}) \cong \tilde{\mathcal{H}}_{G_\alpha^{\tilde{F}}\times G_\beta^{\tilde{F}}}(E_\alpha^{\tilde{F}} \times E_\beta^{\tilde{F}}) \to \tilde{\mathcal{H}}_{G_{\alpha+\beta}^{\tilde{F}}}(E_{\alpha+\beta}^{\tilde{F}})$$

defined as

$$m_{\alpha,\beta}^{\tilde{F}}(g) := v_q^{-\Sigma_{h\in Q_1}\alpha_{s(h)}\beta_{t(h)}-\Sigma_{i\in Q_0}\alpha_i\beta_i}|G_\alpha^{\tilde{F}}\times G_\beta^{\tilde{F}}|^{-1}(p_3)_!(p_2)_!(p_1)^*(g)$$

where if $f: X \to Y$ for $g \in \tilde{\mathcal{H}}_{G_X^{\tilde{F}}}(X)$, $h \in \tilde{\mathcal{H}}_{G_Y^{\tilde{F}}}(Y)$ then $f_!(g)(x) = \Sigma_{y\in f^{-1}(x)}g(y)$ and $f^*(h)(x) = h \circ f(x)$

Comultiplication

$$E_\alpha^{\tilde{F}} \times E_\beta^{\tilde{F}} \xleftarrow{\kappa} F_{\alpha,\beta}^{\tilde{F}} \xrightarrow{\iota} E_{\alpha+\beta}^{\tilde{F}}$$

consider the linear map

$$\delta_{\alpha,\beta}^{\tilde{F}} : \tilde{\mathcal{H}}_{G_{\alpha+\beta}^{\tilde{F}}}(E_{\alpha+\beta}^{\tilde{F}}) \to \tilde{\mathcal{H}}_{G_\alpha^{\tilde{F}} \times G_\beta^{\tilde{F}}}(E_\alpha^{\tilde{F}} \times E_\beta^{\tilde{F}}) \cong \tilde{\mathcal{H}}_{G_\alpha^{\tilde{F}}}(E_\alpha^{\tilde{F}}) \times \tilde{\mathcal{H}}_{G_\beta^{\tilde{F}}}(E_\beta^{\tilde{F}})$$

which maps $r \in \tilde{\mathcal{H}}_{G_{\alpha+\beta}^{\tilde{F}}}(E_{\alpha+\beta}^{\tilde{F}})$ to

$\delta_{\alpha,\beta}^{\tilde{F}}(r) = v_q^{-\Sigma_{h \in Q_1} \alpha_{s(h)} \beta_{t(h)} + \Sigma_{i \in Q_0} \alpha_i \beta_i} \kappa_! \iota^*(r)$

and comultiplication δ is defined as following.

For $r \in \tilde{\mathcal{H}}^{\tilde{F}}(E_\gamma^{\tilde{F}})$

$$\delta(r) = \Sigma_{\alpha+\beta=\gamma} \delta_{\alpha,\beta}^{\tilde{F}}(r)$$

Hall Algebras Via Functions

Theorem

For any isomorphism class $[M]$ of objects in $\mathrm{mod}\, B$ of dimension vector ν, we denote by $\mathcal{O}_M$ the corresponding $\mathbf{G}_\nu^F$-orbits in $\mathbf{E}_\nu^F$ and $1_{\mathcal{O}_M}$ the corresponding characteristic function. The $\mathbb{C}$-linear isomorphism

$$\Phi : \tilde{\mathcal{H}}^{\tilde{F}}(Q) \to \mathcal{H}_q(modB)$$

defined by $1_{\mathcal{O}_M} \mapsto v_q^{\sum_{i \in I} \nu_i^2} u_{[M]}$ preserves the multiplication and the comultiplication.

Now if we could get the Green's formula in sheaves complexes form, then by trace map Green's formula in $\tilde{\mathcal{H}}^{\tilde{F}}(Q)$ is obtained.

Lusztig Induction and Restriction Functors

Consider the map

$$E_\alpha \times E_\beta \xleftarrow{p_1} E' \xrightarrow{p_2} E'' \xrightarrow{p_3} E_\gamma$$

- $E'' = \{(x, W) | x \in E_\gamma,\ x(W) \subset W,\ \underline{\dim}(W) = \beta\}$.
- $E' := \{(x, W, \rho_1, \rho_2) | (x, W) \in E'',\ \rho_1 : V_\gamma / W \cong V_\alpha,\ \rho_2 : W \cong V_\beta$ *are both* $\mathbb{Z}I$ *graded linear homorphism*$\}$.
- $p_1(x, W, \rho_1, \rho_2) = ((\rho_1)_* x|_{V_\gamma / W}, (\rho_2)_* x|_W)$, $p_2(x, W, \rho_1, \rho_2) = (x, W)$ and $p_3(x, W) = x$ for p_1 is smooth of connect fiber and p_2 is a principal bundle.

There exists functor $\mathrm{Ind}^\gamma_{\alpha,\beta} : \mathcal{D}_c(E_\alpha) \times \mathcal{D}_c(E_\beta) \to \mathcal{D}_c(E_\gamma)$ from constructible complexes to constructible complexes:

$$\mathrm{Ind}^\gamma_{\alpha,\beta}(A \boxtimes B) := (p_3)_!(p_2)_\flat(p_1)^*(A \boxtimes B)[d_1 - d_2](\frac{d_1 - d_2}{2})$$

where d_1 is the dimension of the fibre of p_1, and d_2 is the dimension of the fibre of p_2, $d_1 - d_2 = \Sigma_{h \in H} \dim((V_\alpha)_{s(h)}) \dim((V_\beta)_{t(h)}) + \Sigma_{i \in I} \dim((V_\alpha)_i) \dim((V_\beta)_i)$.

Lusztig Induction and Restriction Functors

Now choose a fixed subspace W of V_γ, s.t. $\dim W = \beta$ and there are two $\mathbb{Z}I$ graded linear maps $\rho_1 : V_\gamma / W \cong V_\alpha$ and $\rho_2 : W \cong V_\beta$. For the diagram

$$E_\alpha \times E_\beta \xleftarrow{\kappa} F_{\alpha,\beta} \xrightarrow{\iota} E_{\alpha+\beta}$$

- $F_{\alpha,\beta} = \{x \in E_{\alpha+\beta} | x(W) \subset W\}$, and $F_{\alpha,\beta}$ is a closed subvariety of E_γ.
- κ is defined as $\kappa(x) = ((\rho_1)_* x|_{V_\gamma/W}, (\rho_2)_* x|_W)$, and ι is embedding $\iota(x) = x$.

The restriction functor is defined as follows

$$\mathrm{Res}^{\gamma}_{\alpha,\beta}(C) := \kappa_! \iota^*(C)[-\langle \alpha, \beta \rangle](-\frac{\langle \alpha, \beta \rangle}{2})$$

Remark

- Lusztig proved that $\mathrm{Ind}^{\nu}_{\nu',\nu''} : \mathcal{Q}_{V_{\nu'}} \boxtimes \mathcal{Q}_{V_{\nu''}} \to \mathcal{Q}_{V_\nu}$ and $\mathrm{Res}^{\nu}_{\nu',\nu''} : \mathcal{Q}_{V_\nu} \to \mathcal{Q}_{V_{\nu'}} \boxtimes \mathcal{Q}_{V_{\nu''}}$ by proving

$$\mathrm{Ind}^{\nu}_{\nu',\nu''}(L_{\boldsymbol{\nu}'} \boxtimes L_{\boldsymbol{\nu}''}) = L_{\boldsymbol{\nu}'\boldsymbol{\nu}''}, \tag{1}$$

$$\mathrm{Res}^{\nu}_{\nu',\nu''}(L_{\boldsymbol{\nu}}) = \bigoplus L_{\boldsymbol{\tau}} \boxtimes L_{\boldsymbol{\omega}}[M(\boldsymbol{\tau},\boldsymbol{\omega})](\frac{M(\boldsymbol{\tau},\boldsymbol{\omega})}{2}), \tag{2}$$

  where the direct sum is taken over all flag types $\boldsymbol{\tau},\boldsymbol{\omega}$ satisfying $\boldsymbol{\nu} = \boldsymbol{\tau} + \boldsymbol{\omega}$ and $\sum \tau^l = \nu', \sum \omega^l = \nu''$.
- By Lusztig's results (1), (2), it is easy to check that

$$\mathrm{Res}^{\gamma}_{\alpha',\beta'}\mathrm{Ind}^{\gamma}_{\alpha,\beta}(L_{\boldsymbol{\nu}'} \boxtimes L_{\boldsymbol{\nu}''}) \simeq \bigoplus(\mathrm{Ind}^{\alpha'}_{\alpha_1,\beta_1} \times \mathrm{Ind}^{\beta'}_{\alpha_2,\beta_2})(\tau_\lambda)_!$$
$$((\mathrm{Res}^{\alpha}_{\alpha_1,\alpha_2} L_{\boldsymbol{\nu}'})\boxtimes(\mathrm{Res}^{\beta}_{\beta_1,\beta_2} L_{\boldsymbol{\nu}''}))[-(\alpha_2,\beta_1)](-\frac{(\alpha_2,\beta_1)}{2}),$$

  where the direct sum is taken over all $\lambda = (\alpha_1,\alpha_2,\beta_1,\beta_2) \in \mathcal{N}$.

Mixed Sheaves

For $\mathbb{F}_q$-variety X, and the Frobenius map $Fr : X \to X$, we note that X^F is the point fixed by Fr, first we denote that $D^b_c(X, \bar{Q}_l)$ the category of bounded constructible complexes. And for we have that Fr could induce a functor

$$Fr : D^b_c(X, \bar{Q}_l) \to D^b_c(X, \bar{Q}_l)$$

- A Weil complex is a pair (G, j) s.t. $G \in D^b_c(X, \bar{Q}_l)$ and $j : Fr(G) \to G$ is an isomorphism.
- The Weil sheaf G is called pure of weight ω if the isomorphism j restrict to a closed point x that

  $$j_x : Fr(G)_x \to G_x$$

  s.t. the absolute number of eigenweight of $(j_x)^n$ is $(q^n)^{\frac{\omega}{2}}$ for any $x \in X^{Fr^n}$ and $n \in Z^{>0}$.
- For any Weil complex (F,j), (F,j) is called mixed, if $H^i(F)$ are pure for any i. Let $D^b_m(X)$ is the triangulated category of $D^b_c(X)$ of mixed complexes.

From Sheaves to Hall Algebra

Let $K_m(E_\alpha)$ be the Grothendieck group of $\mathcal{D}^{b,ss}_{G_\alpha,m}(E_\alpha)$($G_\alpha$-equivariant semisimple subcategory of $\mathcal{D}^b_m(E_\alpha)$), and $K_m := \oplus_{\alpha\in\mathbb{N}I} K_m(E_\alpha)$, then the induction functor could induce the multiplication on K_m.

Theorem

There is a surjective algebraic morphism $\chi : K_m \to \mathcal{H}_q(modB)$ which satisfy that $\chi \circ Res = \delta \circ \chi$.

Actually for mixed complex $F = (F, j) \in \mathcal{D}^{b,ss}_{G_\alpha,m}(E_\alpha)$, $\chi(F)$ is a function defined on E_α^{Fr}, for $x \in E_\alpha^{Fr}$

$$\chi(F)(x) = \Sigma_{i\in Z}(-1)^i * tr(j_i, H^i(F)_x)$$

where j_i is the morphism induced by j,

$$j_i : H^i(Fr(F)) \to H^i(F)$$

- $\mathcal{N} = \{\lambda = (\alpha_1, \alpha_2, \beta_1, \beta_2) \in (\mathbb{N}I)^4 | \alpha = \alpha_1 + \alpha_2,\ \beta = \beta_1 + \beta_2,\ \alpha' = \alpha_1 + \beta_1,\ \beta' = \alpha_2 + \beta_2\}$.
- Fix $I-$graded subspace $W^{\beta'} \subset V_\gamma$, $W^{\alpha_2} \subset V_\alpha$, $W^{\beta_2} \subset V_\beta$, with dimension vector β', α_2, β_2.
- $\rho_1^{\beta'} : V_\gamma / W^{\beta_2} \cong V_{\alpha'}$, $\rho_2^{\beta'} : W^{\beta'} \cong V_{\beta'}$, $\rho_1^{\alpha_2} : V_\alpha / W^{\alpha_2} \cong V_{\alpha_1}$, $\rho_2^{\beta_2} : W^{\alpha_2} \cong V_{\alpha_2}$, $\rho_1^{\beta_2} : V_\beta / W^{\beta_2} \cong V_{\beta_1}$, $\rho_2^{\beta_2} : W^{\beta_2} \cong V_{\beta_2}$.

$$E_{\alpha_1} \times E_{\alpha_2} \times E_{\beta_1} \times E_{\beta_2} \xrightarrow{\tau_\lambda} E_{\alpha_1} \times E_{\beta_1} \times E_{\alpha_2} \times E_{\beta_2}$$

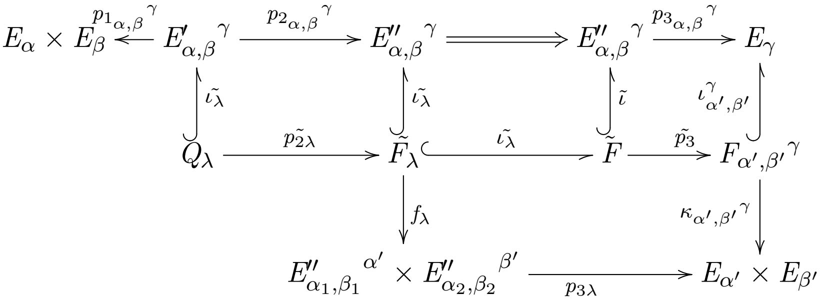
$$(x_{\alpha_1}, x_{\alpha_2}, x_{\beta_1}, x_{\beta_2}) \longmapsto (x_{\alpha_1}, x_{\alpha_2}, x_{\beta_1}, x_{\beta_2})$$

Theorem [Fang, Lan, Xiao]

For any $A \in \mathcal{D}^{b,ss}_{G_\alpha,m}(E_\alpha)$, $B \in \mathcal{D}^{b,ss}_{G_\beta,m}(E_\beta)$ we have that

$$\mathrm{Res}^{\gamma}_{\alpha',\beta'}\mathrm{Ind}^{\gamma}_{\alpha,\beta}(A \boxtimes B) \cong \bigoplus_{\lambda=(\alpha_1,\alpha_2,\beta_1,\beta_2)\in\mathcal{N}} (\mathrm{Ind}^{\alpha'}_{\alpha_1,\beta_1} \times \mathrm{Ind}^{\beta'}_{\alpha_2,\beta_2})(\tau_\lambda)_!$$

$$(\mathrm{Res}^{\alpha}_{\alpha_1,\alpha_2} A \boxtimes \mathrm{Res}^{\beta}_{\beta_1,\beta_2} B)[-(\alpha_2,\beta_1)](-\frac{(\alpha_2,\beta_1)}{2})$$



Prop

The left side of the main theorem is isomorphism to

$$\bigoplus_{\lambda=(\alpha_1,\alpha_2,\beta_1,\beta_2)} (p_{3\lambda})_!(f_\lambda)_!(\tilde{p_2})_b(\tilde{\iota_\lambda})^*(p^\gamma_{1_{\alpha,\beta}})^*(A \boxtimes B)[M](\frac{M}{2})$$

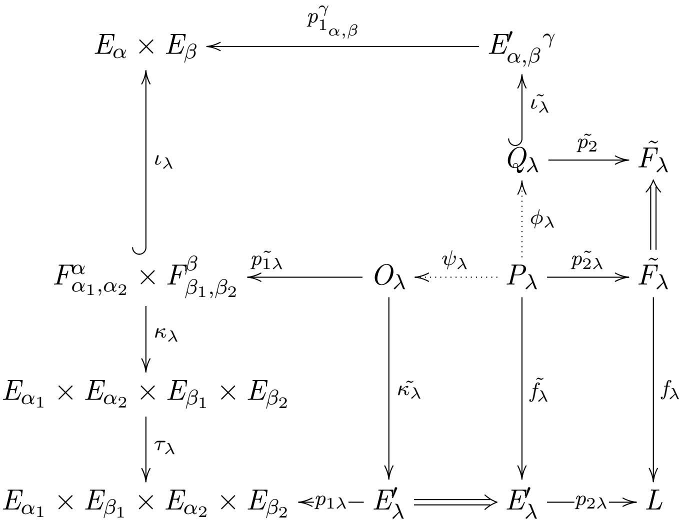
The Left Side

- $\tilde{F}$ in the above graph is defined as $\tilde{F} := E''_{\alpha,\beta}{}^{\gamma} \times_{E_\gamma} F^{\gamma}_{\alpha',\beta'}$; it is a fiber product of $p^{\gamma}_{3_{\alpha,\beta}}$ and $l^{\gamma}_{\alpha',\beta'}$.
- Then we could define the map f_λ. For each $\lambda = (\alpha_1, \alpha_2, \beta_1, \beta_2)$, we define a locally closed subset of $\tilde{F}$: $\tilde{F}_\lambda = \{(x, W) \mid \dim(W \cap W^{\beta'}) = \beta_2\}$. The map $f_\lambda : \tilde{F}_\lambda \to E''_{\alpha_1,\beta_1}{}^{\alpha'} \times E''_{\alpha_2,\beta_2}{}^{\beta'}$ is defined as

$$(x, W) \mapsto (((\rho_1^{\beta'})_* \bar{x}^{W^{\beta'}}, \rho_1^{\beta'}(W/W \cap W^{\beta'})), ((\rho_2^{\beta'})_* x|_{W^{\beta'}}, \rho_2^{\beta'}(W \cap W^{\beta'}))).$$

- Now we denote that $Q_\lambda := E'_{\alpha,\beta}{}^{\gamma} \times_{E^{\gamma}_{\alpha,\beta}{}''} \tilde{F}_\lambda$, which is the fiber product of $p^{\gamma}_{2_{\alpha,\beta}} : E'_{\alpha,\beta}{}^{\gamma} \to E^{\gamma}_{\alpha,\beta}{}''$ and $\tilde{\iota}''_\lambda : \tilde{F}_\lambda \to E''_{\alpha,\beta}{}^{\gamma}$.
- $M = \sum_{h \in H} \alpha_{s(h)} \beta_{t(h)} + \sum_{i \in I} \alpha_i \beta_i - \langle \alpha', \beta' \rangle$.

The Right Side



The Right Side

- $Q_\lambda = \{(x, W, \rho_1, \rho_2) \mid x(V_{\beta'}) \subset V_{\beta'}, W \subset V_\gamma, \dim W = \beta, \dim(W \cap V_{\beta'}) = \beta_2, \rho_1 : V_\gamma / W \cong V_\alpha, \rho_2 : W \cong V_\beta\}$.
- $P_\lambda = \tilde{F}_\lambda \times_L E'_\lambda$, the fiber product of f_λ and $p_{2\lambda}$. O_λ is the pullback of $\kappa_\lambda \circ \tau_\lambda$ and $p_{1\lambda}$. $L := {E''_{\alpha_1,\beta_1}}^{\alpha'} \times {E''_{\alpha_2,\beta_2}}^{\beta'}$.
- There is a smooth morphism ψ_λ with connected fibers of $\dim = L_\lambda - \sum_{h \in H}(\alpha_{1s(h)}\alpha_{2t(h)} + \beta_{1s(h)}\beta_{2t(h)})$, such that $\tilde{f}_\lambda = \tilde{k}_\lambda \psi_\lambda$. And the morphism $\phi_\lambda : P_\lambda \to Q_\lambda$ such that the diagram above commutes.
- $N_\lambda = -\langle \alpha_1, \alpha_2 \rangle - \langle \beta_1, \beta_2 \rangle + \sum_{h \in H}(\alpha_{1s(h)}\beta_{1t(h)} + \alpha_{2s(h)}\beta_{2t(h)}) + \sum_{i \in I}(\alpha_{1i}\beta_{1i} + \alpha_{2i}\beta_{2i})$, $N'_\lambda = N_\lambda - (\alpha_2, \beta_1)$.

Prop

The right side of the theorem is isomorphism to

$$\bigoplus_{\lambda=(\alpha_1,\alpha_2,\beta_1,\beta_2)} (p_{3\lambda})_!(p_{2\lambda})_\flat(\tilde{k}_\lambda)_!(\psi_\lambda)_!(\psi_\lambda)^*(\tilde{p}_{1\lambda})^*(\iota_\lambda)^*(A \boxtimes B)[N'_\lambda]\left(\frac{N'_\lambda}{2}\right)$$

General Case

What if the finite dimensional hereditary algebra B over $k = \mathbb{F}_q$ is not kQ, but the aFr fixed points algebra of (Q, a) ?

For category D, if there is a n-cyclic group $G = \langle a \rangle$ $(a^n = id)$ acting on D, then we could induce a new category $\tilde{D}$.

- object: (B, ϕ) where $B \in D$ and $\phi : aB \to B$ such that the composition of $a^n B \xrightarrow{a^{n-1}\phi} a^{n-1}B \xrightarrow{a^{n-2}\phi} \ldots\ldots \xrightarrow{a\phi} aB \xrightarrow{\phi} B$ is id.
- morphism $f : (B, \phi) \to (B', \phi')$ such that f is morphism in D and $\phi' \circ af = f \circ \phi$.

We consider the category D as $\mathcal{D}^{b,ss}_{G,m}(X)$, and the new category obtained above is denoted as $\widetilde{\mathcal{D}}^{b,ss}_{G,m}(X)$.

The Grothendieck group of $\widetilde{\mathcal{D}^{b,ss}_{G,m}(X)}$ consists of $[(F,\phi)]$ where $[(F,\phi)]$ is the isoclass of $(F,\phi)\in\widetilde{\mathcal{D}^{b,ss}_{G,m}(X)}$ and the relation is as follows.

- if
$$0\to(F,\phi')\to(H,\phi)\to(G,\phi'')\to 0$$
then $[(H,\phi)]=[(F,\phi')]+[G,\phi'']$ and
-
$$[(F,k\phi)]=k[(F,\phi)]$$
-
$$[(F[n],\phi)]=v^{-n}[(F,\phi)]$$
- if (M,ϕ) has the propertity that $M\cong B\oplus a^*B\oplus(a^*)^2B\oplus\ldots\ldots\oplus(a^*)^kB$ and ϕ just maps $a^*((a^*)^{j-1})B$ to $(a^*)^jB$, as a permutation. Then $[(M,\phi)]=0$.

General Case

We now denote that $\tilde{K_m} = \oplus_v \widetilde{K_m(E_v)}$, where $\widetilde{K_m(E_v)}$ is the Grothendieck group of full subcategory of $\widetilde{\mathcal{D}^{b,ss}_{G,m}}(E_v)$ consists of object $((F,j),\phi)$ that $(F, a(j)\circ\phi)$ is mixed sheaf complex under Frobenius map $\tilde{F} := a \circ Fr$

Theorem [Fang, Lan, Wu]

For any $(A,\phi) \in \widetilde{\mathcal{D}^{b,ss}_{G_\alpha,m}}(E_\alpha)$, $(B,\psi) \in \widetilde{\mathcal{D}^{b,ss}_{G_\beta,m}}(E_\beta)$ we have that

$$\mathrm{Res}^{\gamma}_{\alpha',\beta'}\mathrm{Ind}^{\gamma}_{\alpha,\beta}((A,\phi)\boxtimes(B,\psi)) \cong \bigoplus_{\lambda=(\alpha_1,\alpha_2,\beta_1,\beta_2)} (\mathrm{Ind}^{\alpha'}_{\alpha_1,\beta_1} \times \mathrm{Ind}^{\beta'}_{\alpha_2,\beta_2})$$

$$(\tau_\lambda)_!(\mathrm{Res}^{\alpha}_{\alpha_1,\alpha_2}(A,\phi)\boxtimes \mathrm{Res}^{\beta}_{\beta_1,\beta_2}(B,\psi))[-(\alpha_2,\beta_1)](-\frac{(\alpha_2,\beta_1)}{2})$$

up to traceless objects.

Traceless Objects

- For $\mathcal{N}_s$ an $a-$orbit in $\mathcal{N}$, and satisfies that $|\mathcal{N}_s| \geqslant 2$ We set

$$(C_s, \phi_s) = \tilde{p_3}_{s!}\tilde{f}_{s!}\tilde{\iota_s'^*}\tilde{\iota'^*}\tilde{p_2}_\flat\tilde{p}_1^*((A,\varphi)\boxtimes(B,\psi))[M](\frac{M}{2}),$$

  where $C_s = {p_3}_{s!}f_{s!}\iota_s'^*\iota'^*p_{2\flat}p_1^*(A\boxtimes B)[M](\frac{M}{2}), \phi_s : a^*(C_s) \to C_s$ is induced by the isomorphisms $\varphi : a^*(A) \to A, \psi : a^*(B) \to B$ together with $a^*{p_3}_{s!} \cong {p_3}_{s!}a^*, a^*f_{s!} \cong f_{s!}a^*, a^*\iota_s'^* = \iota_s'^*a^*, a^*\iota'^* = \iota'^*a^*, a^*p_{2\flat} \cong p_{2\flat}a^*, a^*p_1^* = p_1^*a^*$.

- The traceless objects above have the form as the direct sum of (C_s, ϕ_s).

Trace Map

Theorem

The trace map $\chi : \tilde{K_m} \to \tilde{\mathcal{H}}^{\tilde{F}}(Q)$
$\chi((((F, j), \phi))(x) = \Sigma_{i \in \mathbb{Z}}(-1)^i * tr((Fr^*(\phi) \circ j)_i, H^i(F)_x)$ for x is $aFr(\tilde{F})-$fixed, which is algebraic surjective morphism and satisfies that $\chi \circ Res \cong \delta \circ \chi$.

This map maps the formula in sheaves form to Green's formula.

Thank you!