Geometric models of graded skew-gentle algebras

Yu Zhou

Based on joint work arXiv:2212.10369v2 with Yu Qiu and Chao Zhang

Tsinghua University

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Motivations

- [Haiden−Katzarkov−Kontsevich 2017] Identify the spaces of stability conditions on derived categories of **graded gentle algebras** with the moduli spaces of quadratic differentials on flat surfaces.
- [Amiot−Opper−Plamondon−Schroll 2018] Complete **derived** invariants of **gentle algebras**, using a geometric model.

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Earlier results for derived categories of (ungraded) skew-gentle algebras:

- [Bekkert–Marcos–Merklen 2003] Classification of **indecomposable objects** in the bounded derived categories of **skew-gentle algebras**.
- [Labardini-Fragoso–Schroll–Valdivieso 2022] and [Amiot 2022] Interpret the classification of indecomposable objects via **geometric terms**.
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Features of our work

Features of our work:

- Consider the perfect derived category of **graded** skew-gentle algebras, which will be useful for the study of silting theory.
- Provide a **basis** for **morphism spaces** (of certain objects) in the perfect derived category using a geometric model, which has not been constructed before, either algebraically or combinatorially.

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Graded quiver

A *graded quiver* is $Q = (Q_0, Q_1, s, t, |\cdot|)$,

- Q_0: vertices;
- Q_1: arrows;
- $s, t : Q_1 \to Q_0$, $s(\alpha) \xrightarrow{\alpha} t(\alpha)$;
- $|\cdot| : Q_1 \to \mathbb{Z}$ a *grading* map

$\mathbf{k}$: a field.

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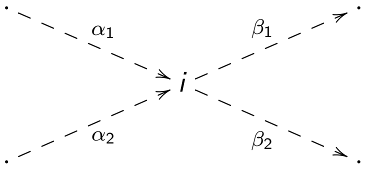

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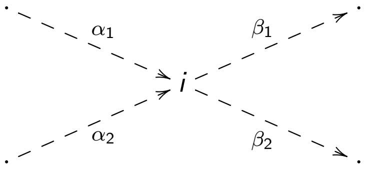

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for some **gentle** algebra $\mathbf{k}Q/\langle R\rangle$ and a subset Sp of loops ε with $\varepsilon^2 \in R$ and $|\varepsilon| = 0$.

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An alternative definition of graded skew-gentle algebra

As introduced in [Burban-Drozd, arXiv:1706.08358], (graded) skew-gentle algebras can be realized as gluings of (graded) path algebras of type A.

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$$(1,1)\xrightarrow{\alpha_1}(1,2)\xrightarrow{\alpha_2}(1,3)\xrightarrow{\alpha_3}(1,4)\xrightarrow{\alpha_4}(1,5)\xrightarrow{\alpha_5}(1,6)\xrightarrow{\alpha_6}(1,7)$$

$$(2,1)\xrightarrow{\beta}(2,2)$$

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with a (symmetric) relation

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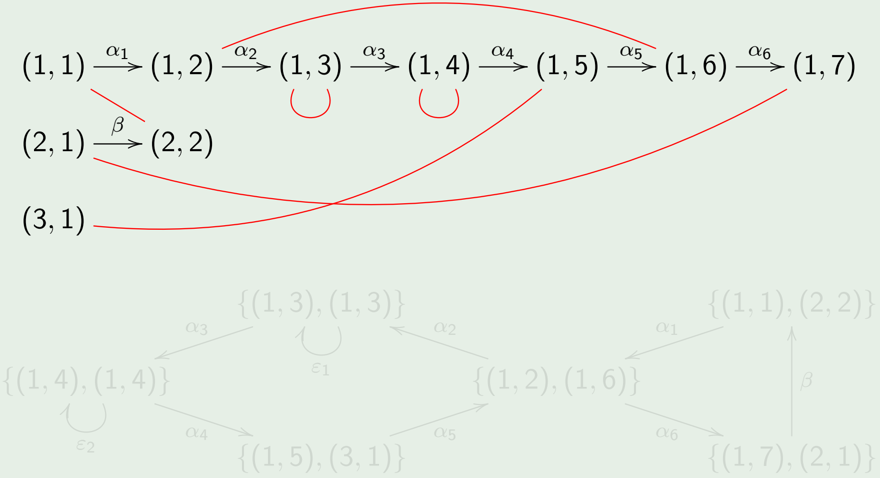

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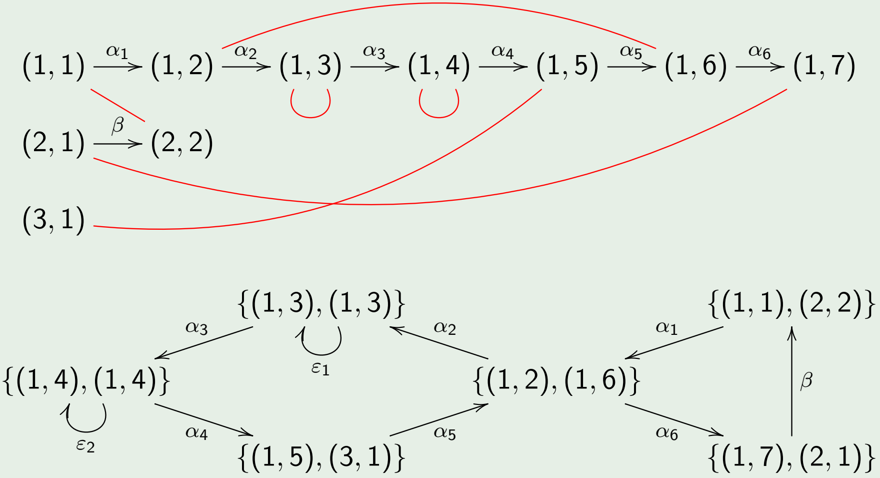

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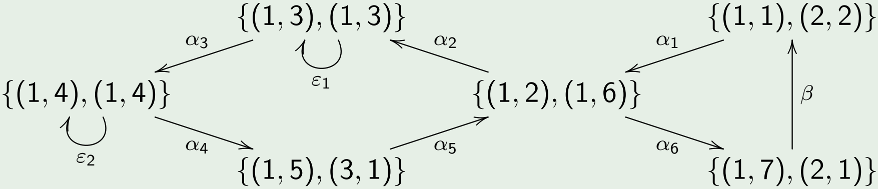

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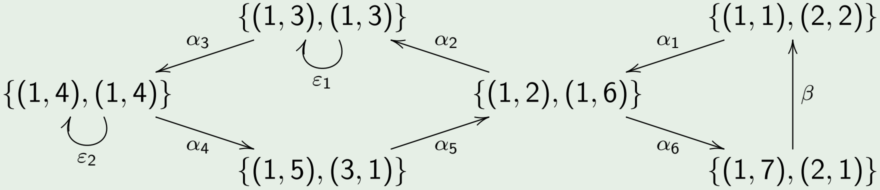

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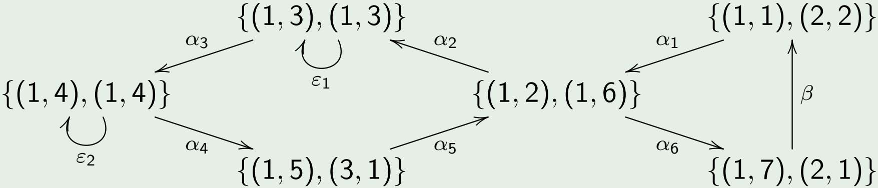

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From gluing of quivers to gluing polygons

Gluing quivers of type $A \rightsquigarrow$ gluing polygons:

1. To each quiver of type A_n, associate an $(n+1)$-gon whose edges (except one) are labeled by the vertices of the quiver anticlockwise.
2. Glue the edges of these polygons following the given relation.
3. For the edge paired with itself, replace it with a once-punctured monogon.

Thus, we get a punctured marked surface with a full formal arc system.

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A geometric description of graded skew-gentle algebras

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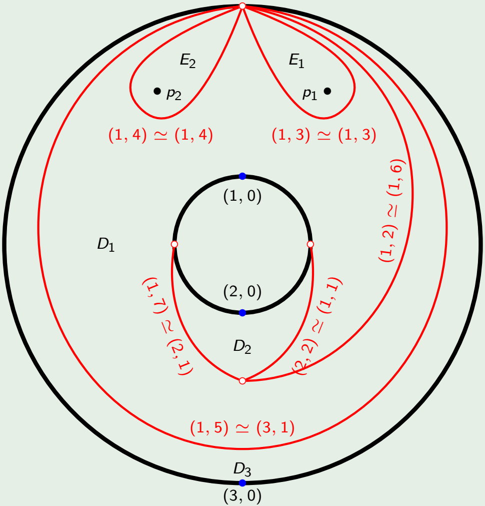

Graded punctured marked surfaces

A *graded punctured marked surface* is $\mathbf{S}^\lambda = (\mathbf{S}, \mathbf{M}, \mathbf{Y}, \mathbf{P}, \lambda)$, where

- $\mathbf{S}$: a connected oriented surface with $\partial\mathbf{S} \neq \emptyset$;
- $\mathbf{M} \subset \partial\mathbf{S}$, $\mathbf{Y} \subset \mathbf{S}$: finite sets of *open* marked points and *closed* marked points, such that each component of $\partial\mathbf{S}$ contains marked points in both $\mathbf{M}$ and $\mathbf{Y}$ alternatively;
- $\mathbf{P} \subset \mathbf{S} \setminus \partial\mathbf{S}$: a finite set of punctures;
- $\lambda : \mathbf{S}^\circ \to \mathbb{P}T(\mathbf{S}^\circ)$ a section of the projectivized tangent bundle of $\mathbf{S}^\circ = \mathbf{S} \setminus (\partial\mathbf{S} \cup \mathbf{P} \cup \mathbf{Y})$, satisfying a certain condition (that the winding number around each puncture is one).

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A *curve* on **S** is an immersion $c : I \to \mathbf{S}$ where $I = [0, 1]$ or S^1.

Definition

A *grading* $\widetilde{c}$ of a curve $c : I \to \mathbf{S}$ is a homotopy class of paths $\widetilde{c}(t)$ in $\mathbb{P}T_{c(t)}(\mathbf{S}^\circ)$ from $\lambda(c(t))$ to $\dot{c}(t)$, varying continuously with $t \in I$.

There are $\mathbb{Z}$ choices of gradings of an underlying curve.

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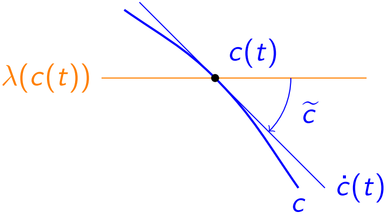

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Grading of curves and intersection index

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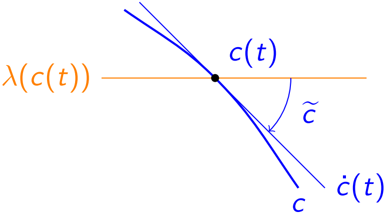

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Intersection index

$\widetilde{c}_1, \widetilde{c}_2$: graded curves.

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The **intersection index** from $\widetilde{c}_1$ to $\widetilde{c}_2$ at intersection $q = c(t_1) = c(t_2)$ is

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where κ is the path in $\mathbb{P}T_q(\mathbf{S}^\circ)$ from $\dot{c}_1(t_1)$ to $\dot{c}_2(t_2)$ given by clockwise rotation by an angle smaller than π.

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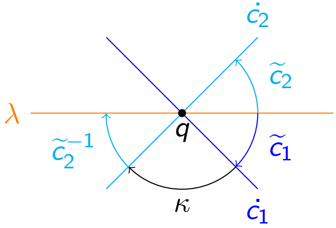

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Full formal closed arc systems

An *arc* is a curve $c : [0,1] \to \mathbf{S}$.

A *full formal* **closed** *arc system* of $\mathbf{S}^\lambda$ is a collection $\mathbf{A}^*$ of graded arcs s.t.

- $\eta(\{0,1\}) \subset \mathbf{Y}$ for any $\widetilde{\eta} \in \mathbf{A}^*$;
- $\eta_1 \cap \eta_2 \cap \mathbf{S}^\circ = \emptyset$, for any $\widetilde{\eta}_1, \widetilde{\eta}_2 \in \mathbf{A}^*$;
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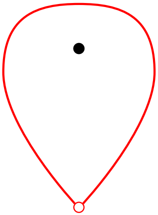
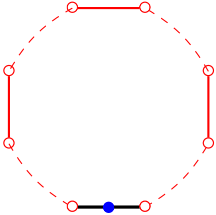
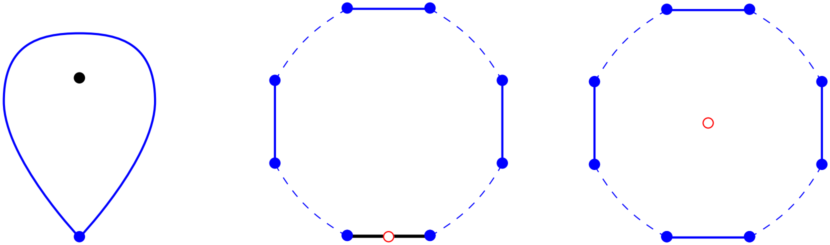




Full formal open arc systems

A *full formal* **open** *arc system* of **S** is a collection **A** of graded arcs s.t.

- $\gamma(\{0,1\}) \subset \mathbf{M}$ for any $\widetilde{\gamma} \in \mathbf{A}$;
- $\gamma_1 \cap \gamma_2 \cap \mathbf{S}^\circ = \emptyset$, for any $\widetilde{\gamma}_1, \widetilde{\gamma}_2 \in \mathbf{A}$;
- **A** cuts out **S** into (**A**-)polygons of the following forms:



Dual between open and closed

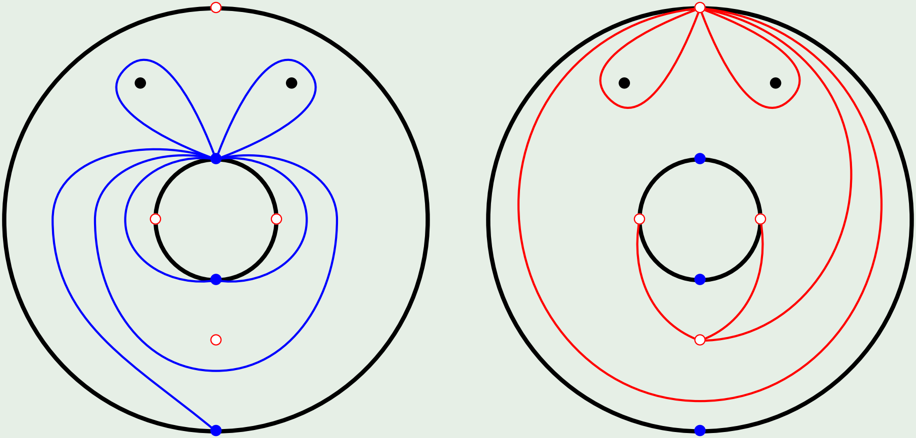
$$\mathbf{A} = \{\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_n\} \overset{\text{dual}}{\longleftrightarrow} \mathbf{A}^* = \{\mathbf{A}^* = \widetilde{\eta}_1, \cdots, \widetilde{\eta}_n\},$$

provided that the following hold.

1. $\widetilde{\gamma}_j$ does not cross $\widetilde{\eta}_i$ for any $j \neq i$.
2. $\widetilde{\gamma}_i$ crosses $\widetilde{\eta}_i$ once and the intersection index is 0, if $\widetilde{\gamma}_i$ does not enclose a puncture.
3. $\widetilde{\gamma}_i$ crosses $\widetilde{\eta}_i$ twice and the intersection indices are 0, if $\widetilde{\gamma}_i$ encloses a puncture.

Example: Dual full formal arc systems

Example



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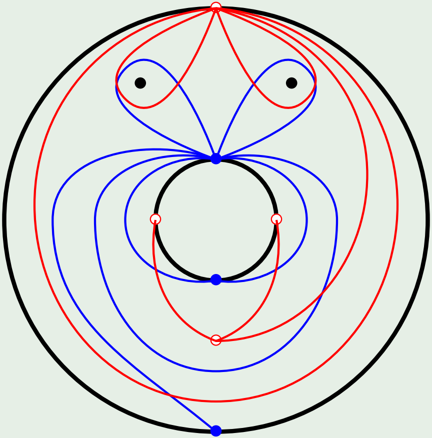

Graded algebras from full formal arc systems

A: a full formal open arc system.

There is an associated triple $(Q_{\mathbf{A}}, R_{\mathbf{A}}, \mathrm{Sp}_{\mathbf{A}})$ as follows:

- $(Q_{\mathbf{A}})_0 = \{1, 2, \cdots, n\}$ indexed by the open arcs $\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_n$ in **A**.
- There is an arrow $\alpha : i \to j$ in $(Q_1)_{\mathbf{A}}$ whenever there is an interior angle of an **A**-polygon having $\widetilde{\gamma}_i$ and $\widetilde{\gamma}_j$ as edges, with $\widetilde{\gamma}_j$ following $\widetilde{\gamma}_i$ in the clockwise order. We take $|\alpha| = i_p(\widetilde{\gamma}_j, \widetilde{\gamma}_i)$, where p is the vertex of the polygon where $\widetilde{\gamma}_i$ and $\widetilde{\gamma}_j$ meet.

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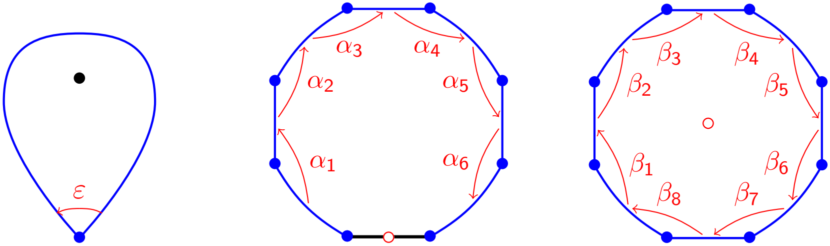

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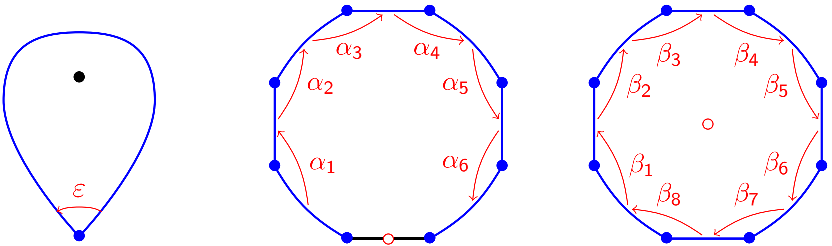
- $R_{\mathbf{A}}$ consists of $\alpha_1\alpha_2$ for $\alpha_1 : i \to j, \alpha_2 : j \to l$ arising from consecutive angles of an **A**-polygon in the clockwise order.



$\varepsilon^2 \in R_{\mathbf{A}}$ $\quad$ $\alpha_i\alpha_{i+1} \in R_{\mathbf{A}}$ $\quad$ $\beta_i\beta_{i+1}, \beta_8\beta_1 \in R_{\mathbf{A}}$

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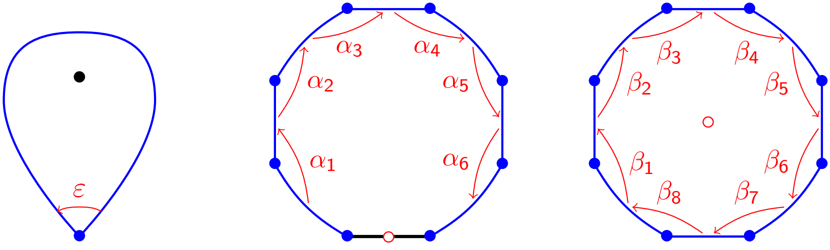

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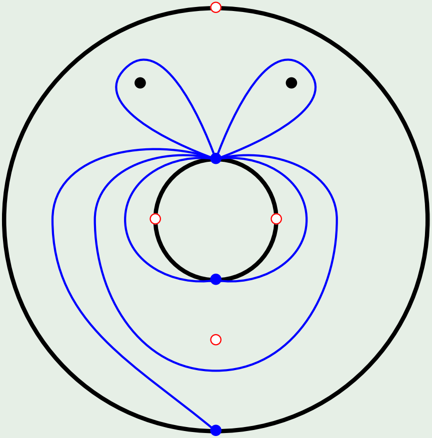
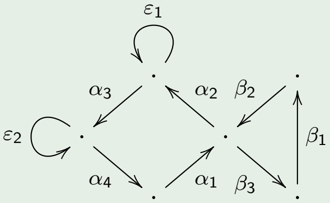
$$\Lambda_{\mathbf{A}} := \mathbf{k}Q_{\mathbf{A}}/\langle R_{\mathbf{A}} \setminus \{\varepsilon^2 \mid \varepsilon \in \mathrm{Sp}_{\mathbf{A}}\} \cup \{\varepsilon^2 - \varepsilon \mid \varepsilon \in \mathrm{Sp}_{\mathbf{A}}\}\rangle.$$

Example



$Q_{\mathbf{A}}$:



$$R_{\mathbf{A}} = \{\alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_4\} \cup \{\beta_1\beta_2, \beta_2\beta_3, \beta_3\beta_1\} \cup \{\varepsilon_1^2, \varepsilon_2^2\}$$

$$\mathsf{Sp} = \{\varepsilon_1, \varepsilon_2\}$$

Geometric realization of graded skew-gentle algebra

Theorem (Qiu–Zhang–Z arXiv:2212.10369v2)

The graded algebras $\Lambda_{\mathbf{A}}$ arising from full formal arc systems $\mathbf{A}$ of graded punctured marked surfaces are exactly the graded skew-gentle algebras.

- The graded gentle case (i.e. $\mathrm{Sp} = \emptyset = \mathbf{P}$) is due to [Lekili–Polishchuk 2018] and [Opper–Plamondon–Schroll 2018].
- For the ungraded (i.e. $|\cdot| \equiv 0$) skew-gentle case, there is another geometric model, using orbifolds [Amiot, Amiot–Brüstle 2022] and [Labardini-Fragoso–Schroll–Valdivieso 2022].

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Perfect categories

We regard a graded skew-gentle algebra Λ as a differential graded algebra (=DGA) with **zero differential**. Denote by

- $\mathcal{D}(\Lambda)$: the derived category of Λ.
- $\operatorname{per}\Lambda$: the smallest thick subcategory of $\mathcal{D}(\Lambda)$ containing Λ.

Definition

A dg Λ-module $M = (|M|, d_M)$ is called *minimal strictly perfect*, if

- there is a decomposition $|M| = \bigoplus_{i=1}^{t} R_i$ for some natural number t, where R_i is a shift of direct summand of Λ, and
- $d_M = (f_{i,j})_{1 \le i,j \le t}$ is a strictly upper triangular matrix, with each entry $f_{i,j} : R_j \to R_i$ in the radical.

$\operatorname{per}^s \Lambda$: full subcategory of $\operatorname{per}\Lambda$ of minimal strictly perfect dg Λ-modules.

Remark (Plamondon 2011, König–Yang 2014)

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A curve $\gamma : I \to \mathbf{S}$ is called *admissible* provided the following hold.

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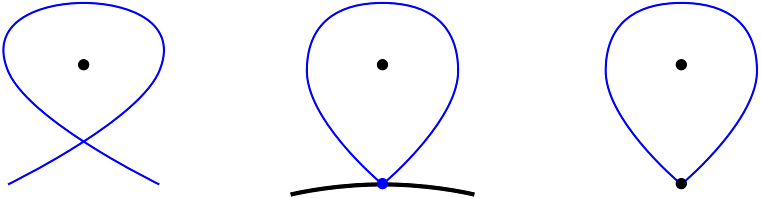

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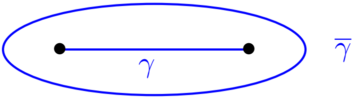

Completion of a curve

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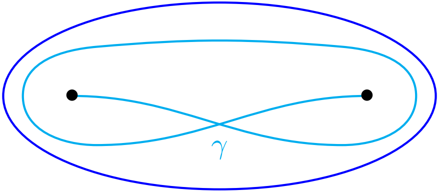

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Graded curves with local system

Definition

A *graded admissible curve with local system* is a pair $(\widetilde{\gamma}, N)$, where

- $\widetilde{\gamma}$ is a graded admissible curve, and
- N is (the isoclass of) an indecomposable A_γ-module, where

$$A_\gamma = \begin{cases} \mathbf{k} & \text{if both endpoints of } \gamma \text{ in } \mathbf{M}, \\ \mathbf{k}[x]/(x^2 - x) & \text{if exactly one endpoint of } \gamma \text{ in } \mathbf{P}, \\ \mathbf{k}\langle x, y\rangle/(x^2 - x, y^2 - y) & \text{if both endpoints of } \gamma \text{ in } \mathbf{P}, \\ \mathbf{k}[x, x^{-1}] & \text{if } \gamma : S^1 \to \mathbf{S}. \end{cases}$$

Classification of indecomposable objects

$\widetilde{\mathrm{OC}}_{\mathrm{l.s.}}(\mathbf{S}^\lambda)$: the set of graded admissible curves with local system.

$\operatorname{ind}\operatorname{per}^s\Lambda$: the set of isoclasses of indecomposable objects in $\operatorname{per}^s\Lambda$.

Theorem (Qiu–Zhang–Z arXiv:2212.10369v2)

There is a bijection

$$X : U \to \operatorname{ind}\operatorname{per}^s\Lambda,$$

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Generalize [Burban-Drozd] to the graded case to obtain a full (but not faithful in general) functor:

$$E : \operatorname{per}^s \Lambda \to \operatorname{rep}(S),$$

which indecomposable-preserving and isomorphism-preserving, where $\operatorname{rep}(S)$ is the category of representations of a bush/clan S.

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Arc objects

Remark

In general, $U \neq \widetilde{\mathrm{OC}}_{\mathrm{l.s.}}(\mathbf{S}^\lambda)$, and we do not have a topological criterion for a graded admissible curve with local system to be in U (and hence, do not give a name for them).

$\widetilde{\mathrm{A1}}(\mathbf{S}^\lambda)$: the set of admissible graded arcs with local system $(\widetilde{\gamma}, N)$ with $\dim N = 1$.

Lemma-Definition.

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Tagged arcs

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There is a natural bijection

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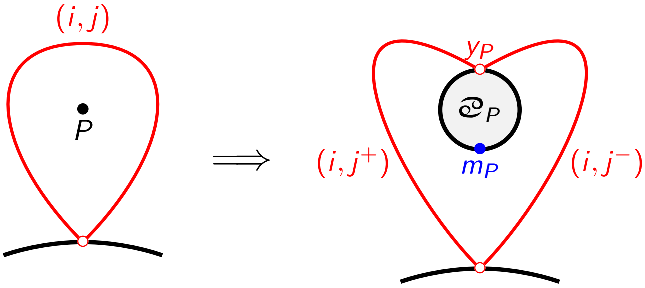
Another geometric model of graded skew-gentle algebras:

- $\mathbf{S}^\lambda \rightsquigarrow \mathbf{S}^\lambda_{☣}$: replace each puncture $P \in \mathbf{P}$ by a boundary component $\mathfrak{D}_P$, called a *binary*, with one open marked point m_P and one closed marked point y_P on it.
- $\mathbf{A}^* \rightsquigarrow \mathbf{A}^*_{☣}$: decompose any edge (i,j) of a once-punctured monogon in $\mathbf{A}^*$ into two graded arcs $(i,j^+),(i,j^-)$ (which inherit the grading from (i,j)).

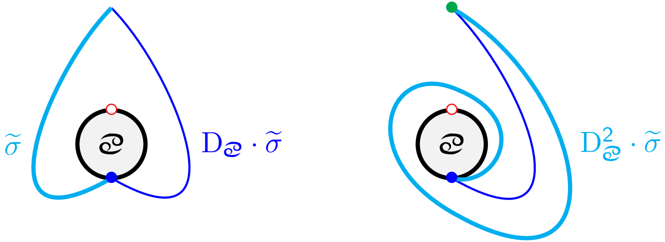


$\mathbb{Z}_2$-symmetry

$\mathrm{D}^2_{☣}$: the subgroup of the usual mapping class group of **S** generated by the squares of Dehn twists $\mathrm{D}_{♋}$ along any binary $♋ \in ☣$.

Definition

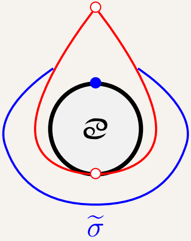
The $\mathrm{D}^2_{☣}$*-orbit* $\mathrm{D}^2_{☣} \cdot \widetilde{\sigma}$ of a graded **admissible** arc $\widetilde{\sigma}$ consists of the graded admissible arcs which are obtained from $\widetilde{\sigma}$ by actions of $\mathrm{D}^2_{☣}$ on the ends (which are in binaries) separately.



Lemma-Definition.

Each $\mathrm{D}^2_{☣}$-orbit of graded admissible arcs contains **exactly one** arc $\widetilde{\sigma}$ satisfying the following equivalent conditions, which is called *unknotted*.

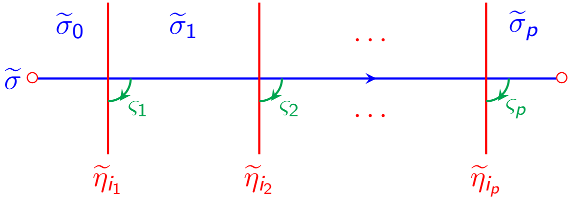
1. $|\widetilde{\sigma} \cap \widetilde{\eta}| \leq |\widetilde{\sigma}' \cap \widetilde{\eta}|$ for any $\widetilde{\sigma}' \in \mathrm{D}^2_{☣} \cdot \widetilde{\sigma}$ and any $\widetilde{\eta} \in \mathbf{A}^*_{☣}$;
2. $\widetilde{\sigma}$ does not have a segment as follows



□

Dg modules associated to unknotted arcs

$\widetilde{\sigma}$: a graded unknotted arc.



Dg Λ-module: $\mathcal{X}^\bullet_{\widetilde{\sigma}} = (|\mathcal{X}^\bullet_{\widetilde{\sigma}}|, d_{\widetilde{\sigma}})$, where

- the underlying graded module $|\mathcal{X}^\bullet_{\widetilde{\sigma}}| = \bigoplus_{1\le l\le p} \Lambda_l[\varsigma_l]$, with $\Lambda_l = \widetilde{\eta}_{i_l}\Lambda$,
- each unbinaried segment $\widetilde{\sigma}_l$ contributes a component of $d_{\widetilde{\sigma}}$ between direct summands $\mathcal{X}^l_{\widetilde{\sigma}}$ and $\mathcal{X}^{l+1}_{\widetilde{\sigma}}$ of $|\mathcal{X}^\bullet_{\widetilde{\sigma}}|$, where

$$\mathcal{X}^l_{\widetilde{\sigma}} = \begin{cases} \Lambda_{l-1} \oplus \Lambda_l & \text{if } \widetilde{\sigma}_{l-1} \text{ is binaried,} \\ \Lambda_l & \text{otherwise.} \end{cases}$$

Classifications of arc objects in the second model

$\widetilde{\mathrm{UC}}(\mathbf{S}^{\lambda}_{☣})$: the set of graded unknotted arcs on $\mathbf{S}^{\lambda}_{☣}$.

$\mathrm{per}^{@}\,\Lambda$: the set of arc objects in $\mathrm{per}\,\Lambda$.

Theorem (Qiu–Zhang–Z arXiv:2212.10369v2)

There is a bijection

$$\mathcal{X}^{\bullet}\colon \widetilde{\mathrm{UC}}(\mathbf{S}^{\lambda}_{☣}) \to \mathrm{per}^{@}\,\Lambda.$$

$\widetilde{\mathrm{TA}}(\mathbf{S}^{\lambda})$: the set of graded tagged arcs on $\mathbf{S}^{\lambda}$.

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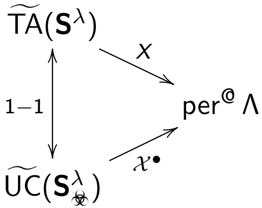
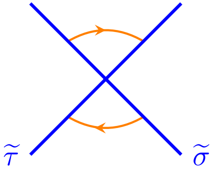
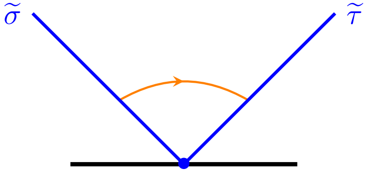


In the left picture, the pair of clockwise angles counts one and in the right picture, the clockwise angle counts one.





$\overrightarrow{\cap}^{\rho}(\widetilde{\sigma},\widetilde{\tau})$: the set of the clockwise angles at intersections from $\widetilde{\sigma}$ to $\widetilde{\tau}$ with index ρ.

Oriented intersection numbers of unknotted arcs

Definition

Let $\widetilde{\sigma}$ and $\widetilde{\tau}$ be two graded unknotted arcs. We define the **intersection number** from $\widetilde{\sigma}$ to $\widetilde{\tau}$ of index ρ to be

$$\overrightarrow{\mathrm{Int}}^{\rho}(\widetilde{\sigma},\widetilde{\tau}) := \min\{|\overrightarrow{\cap}^{\rho}(\widetilde{\sigma}',\widetilde{\tau}')| \mid \widetilde{\sigma}' \in \mathrm{D}^2_{☣} \cdot \widetilde{\sigma},\ \widetilde{\tau}' \in \mathrm{D}^2_{☣} \cdot \widetilde{\tau}\}.$$

$|\overrightarrow{\cap}(\widetilde{\sigma},\widetilde{\tau})| = 2$

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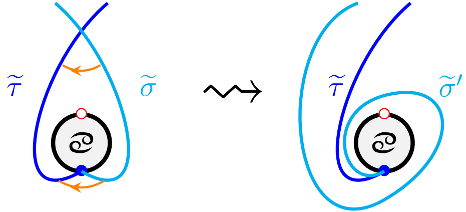

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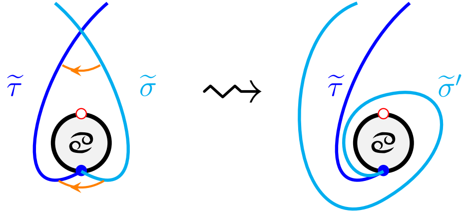

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Intersection-dimension formula

$\widetilde{\sigma}, \widetilde{\tau}$: graded unknotted arcs.

Theorem (Qiu–Zhang–Z arXiv:2212.10369v2)

Let $\widetilde{\sigma}' \in \mathrm{D}^2_{☣} \cdot \widetilde{\sigma}$ *and* $\widetilde{\tau}' \in \mathrm{D}^2_{☣} \cdot \widetilde{\tau}$ *such that* $|\overrightarrow{\cap}^{\rho}(\widetilde{\sigma}', \widetilde{\tau}')| = \overrightarrow{\mathrm{Int}}^{\rho}(\widetilde{\sigma}, \widetilde{\tau})$. *Then there is an explicitly constructed basis* f_q*'s of* $\mathrm{Hom}_{\mathrm{per}\,\Lambda}(\mathcal{X}^{\bullet}_{\widetilde{\sigma}}, \mathcal{X}^{\bullet}_{\widetilde{\tau}}[\rho])$, $q \in \overrightarrow{\cap}^{\rho}(\widetilde{\sigma}', \widetilde{\tau}')$. *In particular,*

$$\overrightarrow{\mathrm{Int}}^{\rho}(\widetilde{\sigma}, \widetilde{\tau}) = \dim \mathrm{Hom}_{\mathrm{per}\,\Lambda}(\mathcal{X}^{\bullet}_{\widetilde{\sigma}}, \mathcal{X}^{\bullet}_{\widetilde{\tau}}[\rho]).$$

- The ungraded gentle case (i.e. $\mathrm{Sp} = \emptyset = \mathbf{P}$ and $|\cdot| = 0$) is due to [Opper-Plamondon-Schroll, 2018] (restricting to arc objects).

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Remarks: Shifts

$\widetilde{c}[1]$: the same underlying arc as $\widetilde{c}$ and the grading is the composition of $\widetilde{c}(t) : \lambda(c(t)) \to \dot{c}(t)$ and the path from $\dot{c}(t)$ to itself given by clockwise rotation by π.

Remark

Since $\mathcal{X}^{\bullet}_{\widetilde{\tau}}[\rho] \cong \mathcal{X}^{\bullet}_{\widetilde{\tau}[\rho]}$, to show the int-dim formula is equivalent to showing

$$\overrightarrow{\mathrm{Int}}^0(\widetilde{\sigma}, \widetilde{\tau}) = \dim \mathrm{Hom}_{\mathrm{per}\,\Lambda}(\mathcal{X}^{\bullet}_{\widetilde{\sigma}}, \mathcal{X}^{\bullet}_{\widetilde{\tau}}).$$

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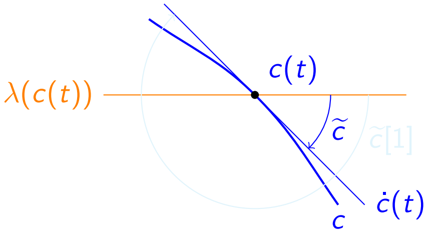

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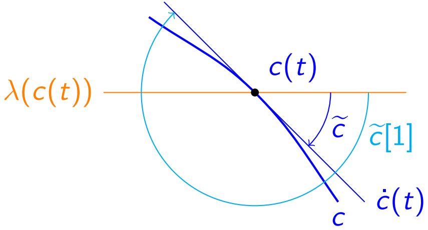

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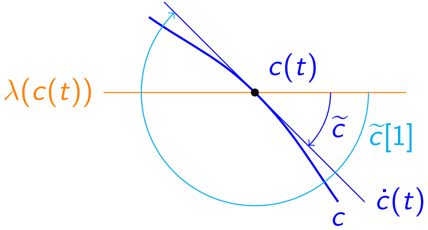

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Remarks: Relationship with bush

Recall there is a full functor

$$E : \mathrm{per}^s \Lambda \to \mathrm{rep}(S).$$

Remark

There is a subset I_1 of $\overrightarrow{\cap}^{\rho}(\widetilde{\sigma}', \widetilde{\tau}')$ such that the images of f_q's, $q \in I_1$, under the functor E are the basis of the morphism space in $\mathrm{rep}(S)$, given in [Geiß 1999].

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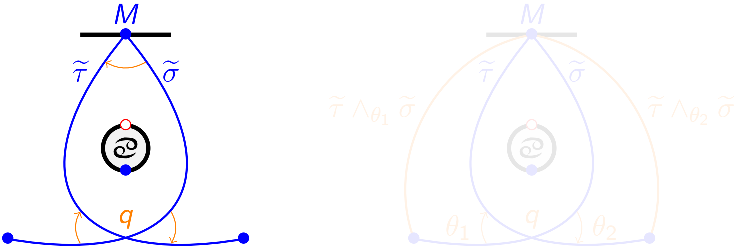
Remarks: Cones of morphisms



$$\mathrm{Cone}(f_q) \cong \mathcal{X}^{\bullet}_{\widetilde{\tau}\wedge_{\theta_1}\widetilde{\sigma}} \oplus \mathcal{X}^{\bullet}_{\widetilde{\tau}\wedge_{\theta_2}\widetilde{\sigma}}.$$

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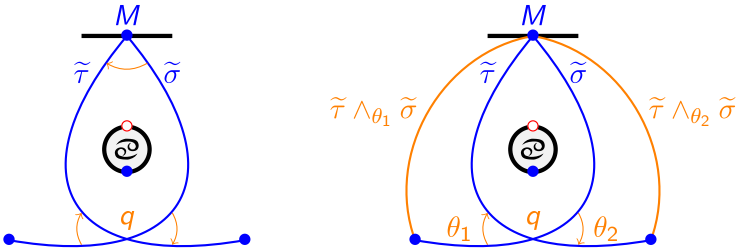

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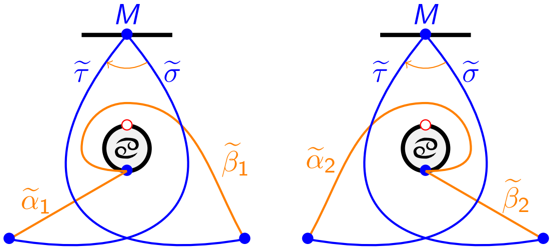

There are two choices for f_M: f'_M and f''_M.

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$$f'_M + f''_M = f_q.$$

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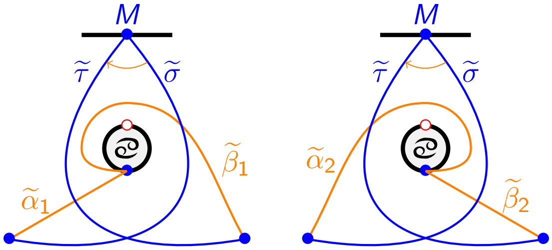

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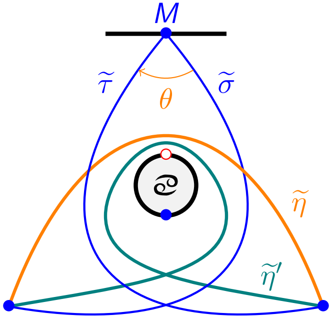
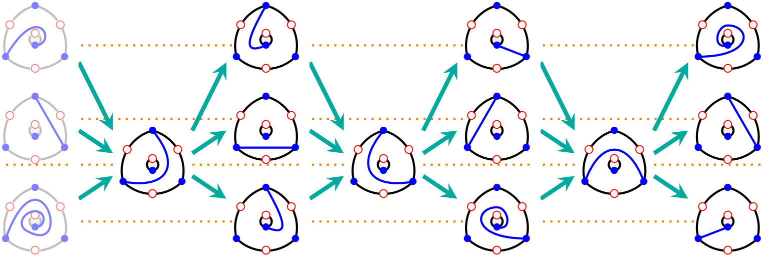
Remarks: Cones of morphisms

$$\mathrm{Cone}(\lambda' f'_M + \lambda'' f''_M) \cong \mathcal{X}^{\bullet}_{\widetilde{\eta}},\ \lambda' \neq 0, \lambda'' \neq 0, \lambda' \neq \lambda''.$$



Here, $\widetilde{\eta}$ is the "admissible version" of the (non-admissible) $\widetilde{\eta}' = \widetilde{\tau}^{\times} \wedge_{\theta} \widetilde{\sigma}^{\times}$.



Thank you for your attention!

Recall that there is a bijection between tagged arcs and unknotted arcs. One can compute the intersection number of the corresponding tagged arcs in the following way