Deformation of 3-Calabi-Yau categories, moduli spaces and Artin braid groups

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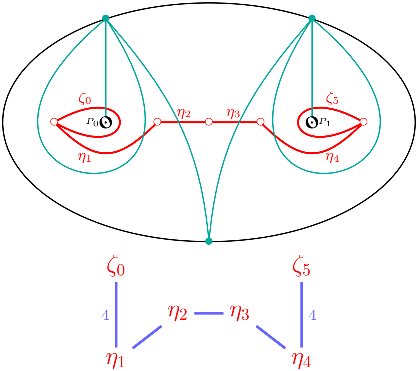
Tsinghua

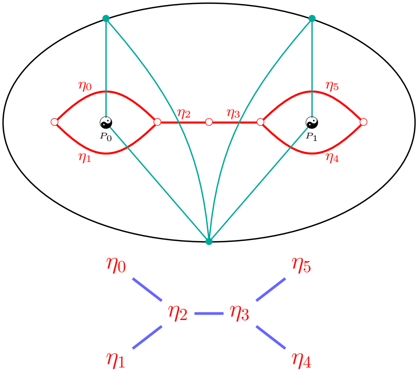
2403.10265

2024.08.08 # ICRA 21^{st}

@ Shanghai Jiao Tong U

Strix ⇜ shape-shifting ⇝ Fox





Outline

1. Deformation of Artin Braid groups
2. Deformation of 3-Calabi-Yau categories
3. Deformation of moduli spaces
4. Deformation of fundamental groups

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Artin braid groups and Weyl groups

Let Υ be a spherical/Euclidean Dynkin diagram.

Definition (Artin briad group associated to Υ)

$$\mathrm{Br}_{\Upsilon} := \langle b_i \mid i \in \Upsilon_0 \rangle \Big/ (\mathrm{Br}^{m_{ij}}(b_i, b_j) \mid \forall i, j \in \Upsilon_0)$$

for $i \xrightarrow{m_{ij}} j$ ($m_{ij} = 2$ if $\nexists$ and omitting $m_{ij} = 3$).

Here $\mathrm{Br}^m(a, b)$: $\underbrace{aba\cdots}_{m} = \underbrace{bab\cdots}_{m}$.

$\mathrm{Co} = \mathrm{Br}^2$: $ab = ba$, $\mathrm{Br} = \mathrm{Br}^3$: $aba = bab$ and Br^4: $abab = baba$.

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Hyperplane arrangements

Let $\mathfrak{h}_\Upsilon$ be some complex space (e.g. the Cartan subalgebra of type Υ) with regular part $\mathfrak{h}_\Upsilon^{\mathrm{reg}}$ (e.g. deleting certain hyperplanes):

$$\mathfrak{h}_\Upsilon^{\mathrm{reg}} :=\mathfrak{h}_\Upsilon \setminus \Big(\bigcup_\alpha H_\alpha\Big).$$

It is well-known that the Weyl group W_Υ acts on $\mathfrak{h}^{\mathrm{reg}}$ freely with

$$\pi_1(\mathfrak{h}_\Upsilon^{\mathrm{reg}}/W_\Upsilon) = \mathrm{Br}_\Upsilon .$$

Moreover, such a quotient space is $K(\pi,1)$:

Theorem (Deligne, Paolini-Salvetti)

The universal cover of $\mathfrak{h}^{\mathrm{reg}}/W_\Upsilon$ is contractible.

Remark: Kontsevich-Zorich conjecture that 'Each connected component of the (strata of) moduli spaces of quadratic differentials is $K(\pi,1)$.

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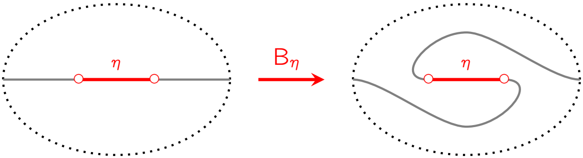

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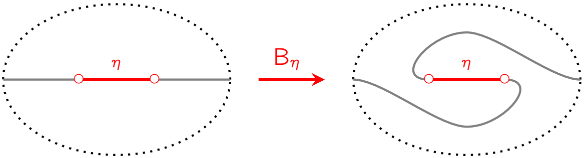

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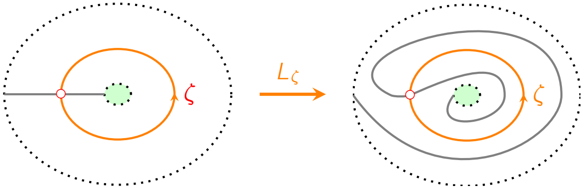
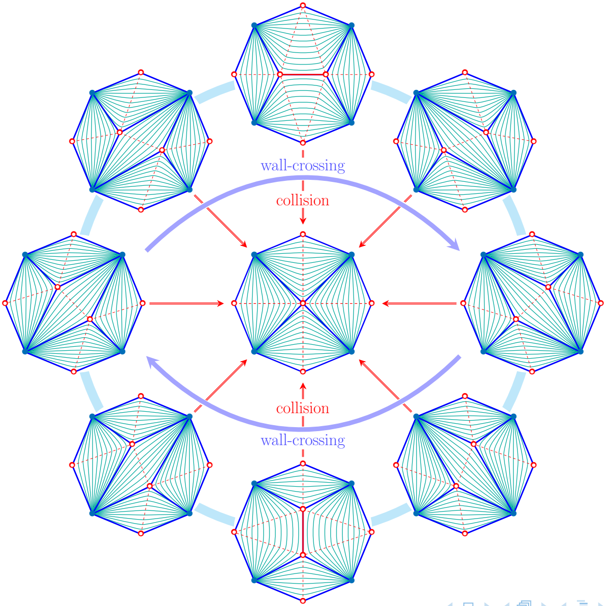


Geometric generator: braid twist



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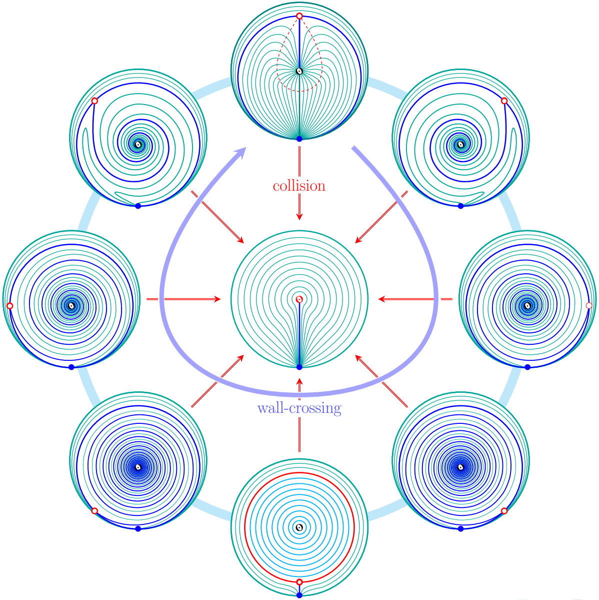

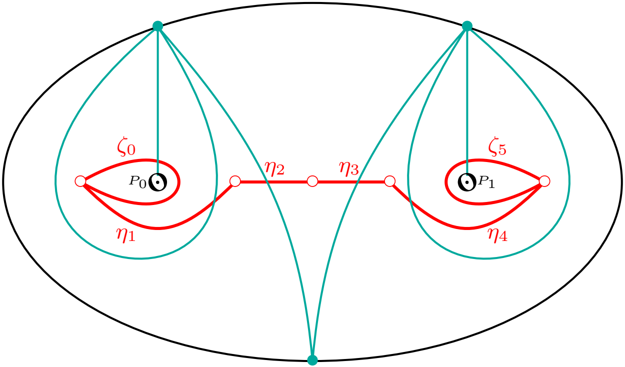
ζ_0
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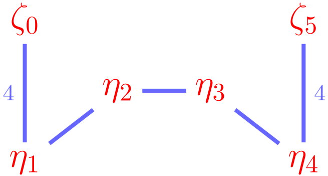
η_2
η_3
η_4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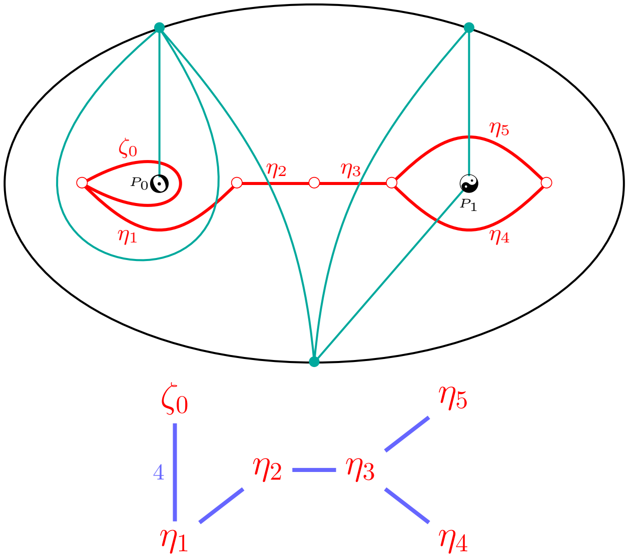
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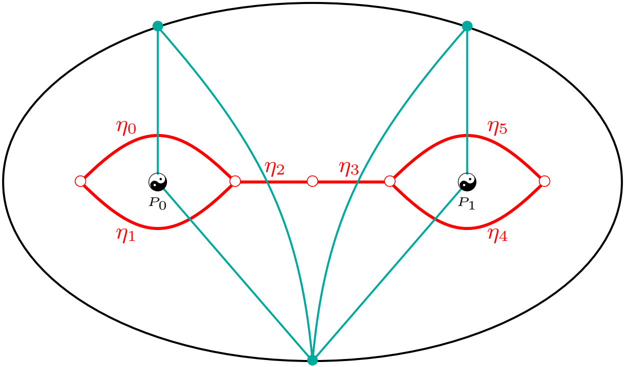
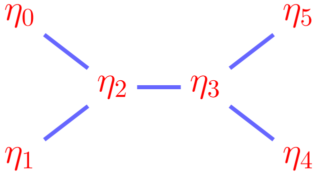
Affine type D Fox





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Deformed N-Calabi-Yau completion (Keller)

Let A be a dg algebra.
Let $A^e = A \otimes A^{op}, \theta = \mathrm{RHom}_{A^e}(A, A^e)$.

The N-Calabi-Yau completion of A $\Pi_N(A)$ is

$$T_A(\Theta) = A \oplus \Theta \oplus (\Theta \otimes_A \Theta) \oplus \cdots$$

for $\Theta = \theta[N-1]$.

Given

$$c \in \mathrm{HH}_{N-1}(A) \cong \mathrm{Hom}_{\mathcal{D}(A^e)}(\Theta, A[1]).$$

The deformed N-Calabi-Yau completion is $\Pi_N(A, c)$ by adding c into the differential of $\Pi(A)$. The perfectly-valued/f.d. derived category $\mathrm{pvd}(\Pi_N(A, c))$ is N-Calabi-Yau.

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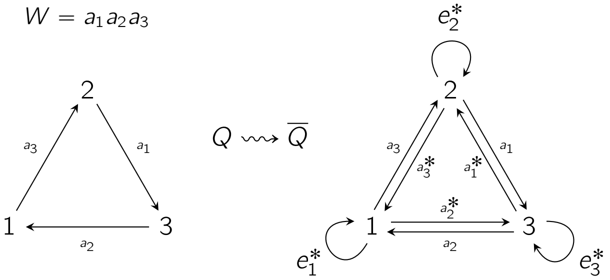
Deformation of 3-CY categories via QP

Given a quiver with potential (Q, W). Let $A = \mathbf{k}\overline{Q}$ and $c = c(W)$.
There is a 3-CY cat. $\mathcal{D}_3(Q, W) = \mathrm{pvd}(\Gamma(Q, W)) \cong \mathrm{pvd}(\Pi_3(A, c))$.



Diff.: $\begin{cases} \sum_{i \in Q_0} de_i^* = \sum_{a \in Q_1} [a, a^*], \\ da^* = \partial_a W. \end{cases}$

e.g. $\begin{cases} de_i = a_{j-1} a_{j+1}^* - a_{j+1}^* a_{j-1}, \\ da_j^* = a_{j+1} a_{j-1}. \end{cases}$

QPs associated to triangulations

Given a triangulation T of $\mathbf{S}^{\odot}$ and a partition $\odot = ☾ \cup ☽$ of the set of punctures.
waning vs. waxing Crescent. Turn waxing Crescent into vortex (YinYang)

The quiver is $Q_{\mathrm{T}}^{☾,☯}$=(arcs, angles).

The potential is $W_{\mathrm{T}}^{☾,☯} = \sum_{T\in\mathrm{T}} W_T - \sum_{V\in ☽} W_V$.

$$\mathcal{D}_3(\mathbf{S}^{☾,☯}) = \mathcal{D}_3(Q_{\mathrm{T}}^{☾,☯}, W_{\mathrm{T}}^{☾,☯}).$$

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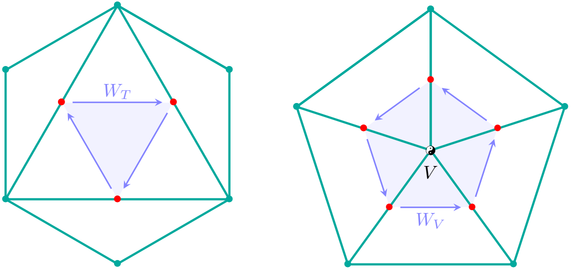

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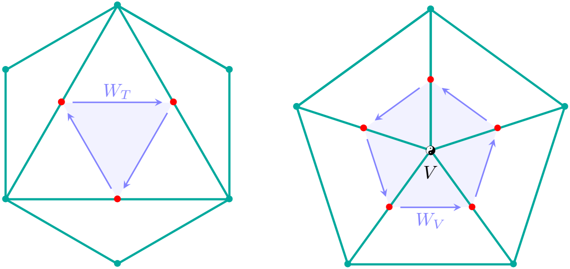

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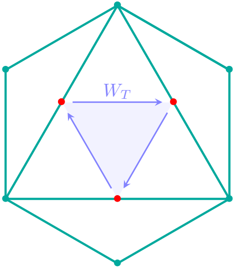

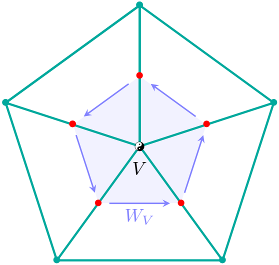

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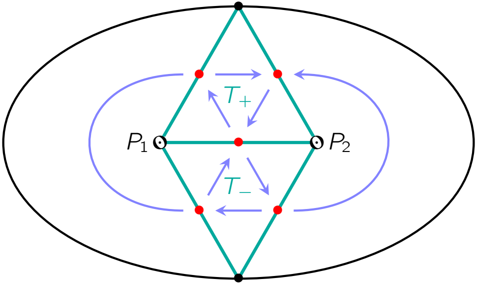
Examples: QPs



- (☾, ☽) = (☉, ∅) and $W_{\mathrm{T}}^{☉} = W_{T_+} + W_{T_-}$.
- (☾, ☽) = (P_1, P_2) and $W_{\mathrm{T}}^{☾,☯} = W_{T_+} + W_{T_-} - W_{P_1}$.
- (☾, ☽) = (∅, ☉) and $W_{\mathrm{T}}^{☯} = W_{T_+} + W_{T_-} - W_{P_1} - W_{P_2}$.

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Stability conditions on triangulated categories

A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ =

- a central charge Z &
- a slicing, i.e. a $\mathbb{R}$-collection of t-structures,

satisfying contain condition.

Theorem (Bridgeland)

Stab $\mathcal{D}$ *is a* $\mathbb{C}$*-manifold with local coordinate* $Z \in \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$.

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Stab as Quad (CHQ vs. Q vs. BS)

Theorem (Q)

There is an isomorphism between complex orbifolds

$$\frac{\mathrm{FQuad}^{\pm}(\mathbf{S}^{☾,☯})}{\mathrm{MCG}(\mathbf{S}^{☾,☯})} \cong \frac{\mathrm{Stab}^{\circ}(\mathcal{D}_3(\mathbf{S}^{☾,☯}))}{\mathrm{Aut}^{\circ}\,\mathcal{D}_3(\mathbf{S}^{☾,☯})},$$

which specializes to

- *Bridgeland-Smith for* $(☾, ☯) = (\varnothing, ⊙)$.
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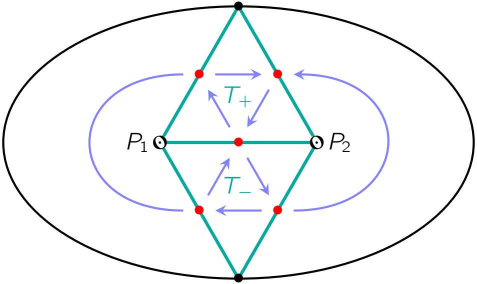
Foliation = Slicing
Period map = Central charge &
Broken geodesic = HN-filtration.

Examples: QPs



$(☾, ☽) = (\odot, \varnothing)$ and $W_{\mathrm{T}}^{\odot} = W_{T_+} + W_{T_-}$:

$$\mathrm{Stab}^\circ(\mathcal{D}_3(\mathrm{S}_\Delta^{\odot}))/\,\mathrm{Aut} \cong \quad \mathrm{Quad}(0; 1^4, (-2)^2, -4).$$

$(☾, ☽) = (P_1, P_2)$ and $W_{\mathrm{T}}^{☾,☯} = W_{T_+} + W_{T_-} - W_{P_1}$:

$$\mathrm{Stab}^\circ(\mathcal{D}_3(\mathrm{S}^{☾,☯}))/\,\mathrm{Aut} \cong \cdots \bigsqcup \mathrm{Quad}(0; 1^3, -1, -2, -4)^{\mathbb{Z}_2}.$$

$(☾, ☽) = (\varnothing, \odot)$ and $W_{\mathrm{T}}^{☯} = W_{T_+} + W_{T_-} - W_{P_1} - W_{P_2}$:

$$\mathrm{Stab}^\circ(\mathcal{D}_3(\mathrm{S}^{☯}))/\,\mathrm{Aut} \cong \cdots \bigsqcup \mathrm{Quad}(0; 1^2, (-1)^2, -4)^{\mathbb{Z}_2^2}.$$

Contents

1 Deformation of Artin Braid groups

2 Deformation of 3-Calabi-Yau categories

3 Deformation of moduli spaces

4 Deformation of fundamental groups

Affine type CBD hyperplane arrangements

Let

$$V_0 = \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{C}^n \mid x_j \neq \pm x_k, \forall j \neq k\} \tag{1}$$

and

$$\begin{cases} V_{\widetilde{C_n}} = \{\mathbf{x} \in V_0 \mid \forall x_j \notin \frac{1}{2}\mathbb{Z}\}, \\ V_{\widetilde{B_n}} = \{\mathbf{x} \in V_0 \mid \forall x_j \notin \mathbb{Z}\}, \\ V_{\widetilde{D_n}} = V_0. \end{cases} \tag{2}$$

Then there are homotopy equivalences

$$V_\Upsilon / W_\Upsilon \simeq \mathfrak{h}_\Upsilon^{\mathrm{reg}} / W_\Upsilon$$

for $\Upsilon = \widetilde{C_n}, \widetilde{B_n}$ or $\widetilde{D_n}$.

The explicit map

Consider the following map

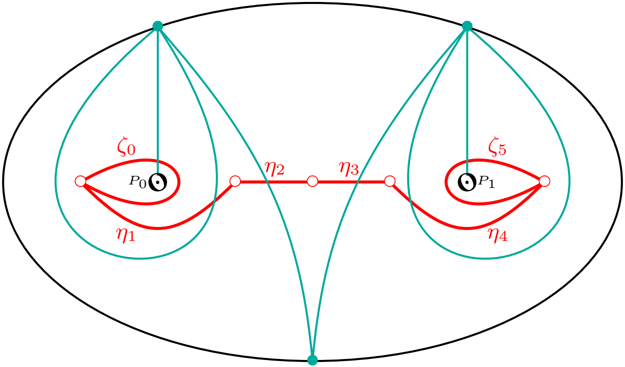
$$\begin{array}{ccccc} \iota\colon \mathbb{C} & \xrightarrow{e^{2\pi \mathrm{i}(?)}} & \mathbb{C}^* & \xrightarrow{\eta} & \mathbb{C}^{0,1}, \\ x & \mapsto & y & \mapsto & \frac{1}{4}(1-y)(1-\frac{1}{y}), \end{array}$$

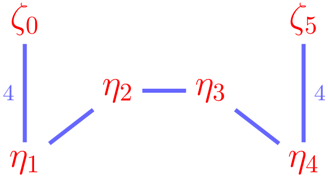
where $\mathbb{C}^{0,1} = \mathbb{C} \setminus \{0,1\}$ and η is a branched double cover.
It induces a homotopy equivalence

$$V_{\widetilde{C_n}}/W_{\widetilde{C_n}} \simeq \mathrm{conf}_n(\mathbb{C}^{0,1}), \tag{3}$$

where $\mathfrak{S}_n$ is permutation group of n elements.

Affine type C Strix





FQuad($\mathbf{S}^{\odot}$) consists of differentials of the form (with a ZZ-framing)

$$\phi(z) = \lambda \cdot \frac{\prod_{i=1}^{n}(z-z_i)}{z^2(z-1)^2} dz^2, \quad \lambda \in \mathbb{C}^*, z_i \in \mathbb{C}^{0,1}, \quad z_i \neq z_j.$$

Hence, FQuad($\mathbf{S}^{\odot}$) is homotopy equivalent to $\mathrm{conf}_n(\mathbb{C}^{0,1})$:

$$\mathrm{FQuad}(\mathbf{S}^{\odot}) \xrightarrow{\simeq} \mathrm{conf}_n(\mathbb{C}^{0,1}) \simeq \mathfrak{h}^{\mathrm{reg}}_{\widetilde{C_n}} / W_{\widetilde{C_n}}. \tag{4}$$

The contractible space $\mathrm{Stab}^\circ(\mathcal{D}_3(\mathbf{S}^{\odot}_\Delta)) \cong \mathrm{FQuad}^{\mathbb{T}}(\mathbf{S}^{\odot}_\Delta)$ is the universal cover of

$$\mathrm{FQuad}(\mathbf{S}^{\odot}) = \mathrm{FQuad}(0; 1^n, (-2)^2, -n-2),$$

with covering group

$$\pi_1 \mathrm{FQuad}(\mathbf{S}^{\odot}) = \mathrm{Br}(\widetilde{C_n}).$$

FQuad($\mathbf{S}^{\mho}$) consists of differentials of the form (with a ZZ-framing)

$$\phi(z) = \lambda \cdot \frac{\prod_{i=1}^{n}(z - z_i)}{z^2(z-1)^2} dz^2, \quad \lambda \in \mathbb{C}^*, z_i \in \mathbb{C}^{0,1}, \quad z_i \neq z_j.$$

Hence, FQuad($\mathbf{S}^{\mho}$) is homotopy equivalent to $\mathrm{conf}_n(\mathbb{C}^{0,1})$:

$$\mathrm{FQuad}(\mathbf{S}^{\mho}) \xrightarrow{\simeq} \mathrm{conf}_n(\mathbb{C}^{0,1}) \simeq \mathfrak{h}^{\mathrm{reg}}_{\widetilde{C_n}} / W_{\widetilde{C_n}}. \tag{4}$$

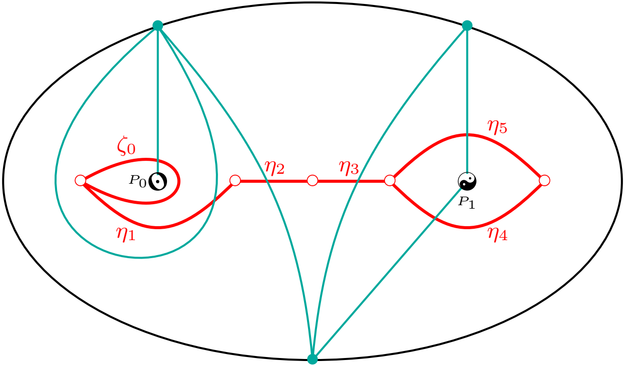
The contractible space $\mathrm{Stab}^\circ(\mathcal{D}_3(\mathbf{S}^{\mho}_\Delta)) \cong \mathrm{FQuad}^{\mathbb{T}}(\mathbf{S}^{\mho}_\Delta)$ is the universal cover of

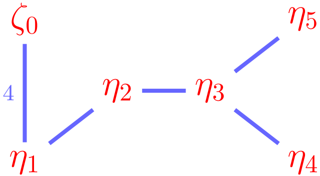
$$\mathrm{FQuad}(\mathbf{S}^{\mho}) = \mathrm{FQuad}(0; 1^n, (-2)^2, -n-2),$$

with covering group

$$\pi_1 \mathrm{FQuad}(\mathbf{S}^{\mho}) = \mathrm{Br}(\widetilde{C_n}).$$

Affine type B Mix





Affine type B Mix

Partial compactification with orbifolding = adding back certain hyperplanes (in the covering).

Hence:

$$\mathrm{FQuad}(\mathbf{S}^{\text{☾},\text{☯}}) \xrightarrow{\simeq} \overline{\mathrm{conf}_n(\mathbb{C}^{0,1})}^{1} \simeq \mathfrak{h}^{\mathrm{reg}}_{\widetilde{B_n}} / W_{\widetilde{C_n}}. \tag{5}$$

and then

$$\mathrm{FQuad}^{\pm}(\mathbf{S}^{\text{☾},\text{☯}}) \simeq \mathfrak{h}^{\mathrm{reg}}_{\widetilde{B_n}} / W_{\widetilde{B_n}}.$$

with

$$\pi_1 \, \mathrm{FQuad}^{\pm}(\mathbf{S}^{\text{☾},\text{☯}}) = \mathrm{Br}(\widetilde{B_n}). \tag{6}$$

Partial compactification with orbifolding = adding back certain hyperplanes (in the covering).

Hence:

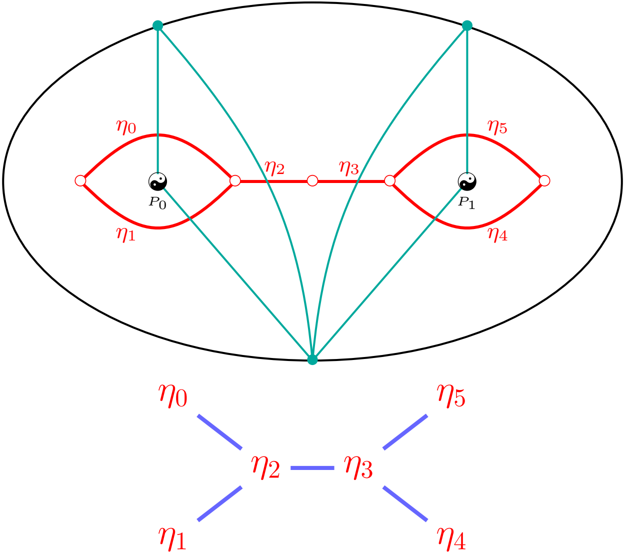
$$\mathrm{FQuad}(\mathbf{S}^{\complement,☯}) \xrightarrow{\simeq} \overline{\mathrm{conf}_n(\mathbb{C}^{0,1})}^{1} \simeq \mathfrak{h}^{\mathrm{reg}}_{\widetilde{B}_n}/W_{\widetilde{C}_n}. \tag{5}$$

and then

$$\mathrm{FQuad}^{\pm}(\mathbf{S}^{\complement,☯}) \simeq \mathfrak{h}^{\mathrm{reg}}_{\widetilde{B}_n}/W_{\widetilde{B}_n}.$$

with

$$\pi_1\,\mathrm{FQuad}^{\pm}(\mathbf{S}^{\complement,☯}) = \mathrm{Br}(\widetilde{B}_n). \tag{6}$$



η_0
η_1
η_2
η_3
η_4
η_5
P_0
P_1

Similarly,

$$\mathrm{FQuad}(\mathbf{S}^{\yinyang}) \xrightarrow{\simeq} \overline{\mathrm{conf}_n(\mathbb{C}^{0,1})}^{0,1} \simeq \mathfrak{h}_{\widetilde{D_n}}^{\mathrm{reg}}/W_{\widetilde{C_n}}. \tag{7}$$

and then

$$\mathrm{FQuad}^{\pm}(\mathbf{S}^{\yinyang}) \simeq \mathfrak{h}_{\widetilde{D_n}}^{\mathrm{reg}}/W_{\widetilde{D_n}}.$$

with

$$\pi_1 \, \mathrm{FQuad}^{\pm}(\mathbf{S}^{\yinyang}) = \mathrm{Br}(\widetilde{D_n}). \tag{8}$$

Summary

Υ: any non-exceptional spherical/Dynkin diagram; $\exists$ a DMSx $\mathbf{S}^{☾,☯}$ such that:

Theorem (Q)

There is a homotopy equivalence

$$\mathrm{FQuad}^{\pm}(\mathbf{S}^{☾,☯}) \simeq \mathfrak{h}_{\Upsilon}^{\mathrm{reg}}/W_{\Upsilon}$$

whose fundamental group fits into the following commutative diagram:

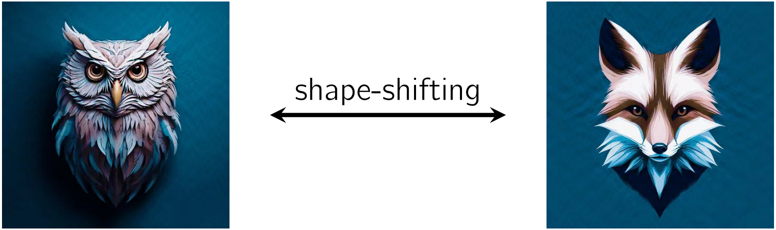
$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Br}(\Upsilon) & \longrightarrow & \mathrm{SBr}(\mathbf{S}^{☾,☯}) & \xrightarrow{\mathrm{AJ}^{☯}} & \mathbf{H}_1(\mathbf{S}^{☯}) & \longrightarrow & 1 \\
 & & \| & & \| & & \| & & \\
1 & \longrightarrow & \pi_1\,\mathrm{FQuad}^{\pm}(\mathbf{S}^{☾,☯}) & \longrightarrow & \pi_1\,\mathrm{conf}_{\Delta}(\mathbf{S}^{☾,☯}) & \longrightarrow & \mathbf{H}_1(\mathbf{S}) \oplus \mathbb{Z}_2^{\oplus ☯} & \longrightarrow & 1.
\end{array}$$

In the 'lattice' case (i.e. spherical, $\widetilde{A}_n$ or $\widetilde{C}_n$), a contractible universal cover of $\mathfrak{h}_{\Upsilon}^{\mathrm{reg}}/W_{\Upsilon}$ is given by $\mathrm{Stab}^{\circ}(D_3(\mathbf{S}_{\Delta}^{☾,☯}))$.

The Comparison: Strix-Mix-Fox as C-B-D

POV	DMSp $\mathbf{S}_\Delta^{\odot}$	DMSx $\mathbf{S}_\Delta^{☾,☯}$	DMSv $\mathbf{S}_\Delta^{☯}$
Moduli Space Dynamical System	Natural	Natural∼ish	Unnatural
Quiver Algebras Rep. Theory	Natural Gentle/inf.d.	soso	Natural Skew-gentle/f.d.
Quiver w/ Potential Cluster theory	Unnatural Degenerate	Unnatural	Natural Non-degenerate
Perverse Schobers Higher cat.	Natural	-	-
DT-invariants Math. Physics	Natural	?	Natural?
Hyperplane Arrangements	Natural	Natural	Natural

Thank you!



∃ correspondences:

- Alg: Change of twist groups.
- Cat: Deformation of 3-Calabi-Yau categories.
- Geo: Partial compactification with orbifolding of moduli spaces.
- Top: Taking sub-quotient of mapping class groups (as π_1).