A generalization of Dugas' construction on stable auto-equivalences for symmetric algebras

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joint work with Nengqun Li

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Notations

- k: a field
- A: a finite-dimensional self-injective k-algebra
- $\mathrm{mod}\text{-}A$: the category of finite-dimensional right A-modules
- $\underline{\mathrm{mod}}\text{-}A$: the **stable category** of $\mathrm{mod}\text{-}A$ by factoring out the morphisms that factor through a projective A-module
- $D^b(\mathrm{mod}\text{-}A)$: the **bounded derived category** of $\mathrm{mod}\text{-}A$
- $A^e = A^{op} \otimes_k A$: the **enveloping algebra** of A
    - $\mathrm{lrp}(A)$: the subcategory of $\mathrm{mod}\text{-}A^e$ consisting of left-right projective A^e-modules
    - $\underline{\mathrm{lrp}}(A)$: the stable category of $\mathrm{lrp}(A)$ obtained by factoring out the morphisms that factor through a projective A^e-module

Morita equivalence

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- By classical Morita theory, A and B are Morita equivalent if and only if there exists a projective generator P_A in mod-A such that the endomorphism algebra $\mathrm{End}(P_A)$ is isomorphic to B.
- In this case, the tensor functor $- \otimes_B P_A$: mod-$B \to$ mod-A is an equivalence of abelian categories.

Derived equivalence

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- By Rickard's Morita theory for derived categories, one can construct a derived equivalence by using the (one-sided) tilting complex.
- Moreover, if A and B are derived equivalent, then there exists a derived equivalence $D^b(\text{mod-}B) \to D^b(\text{mod-}A)$ given by the derived tensor functor of some two-sided tilting complex.

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- The projective modules are not visible in this category and there is no substitute known in $\underline{\mathrm{mod}}$-A for projective generators in mod-A or tilting complexes in $D^b(\mathrm{mod}\text{-}A)$.
- An analogue of Morita theory for stable categories is missing.

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- **Auslander-Reiten conjecture**: If two finite dimensional k-algebras A and B are stably equivalent, then the number of isomorphism classes of non-projective simple modules over A and B are the same.
- Martinez-Villa: AR-conjecture is reduced to stable equivalences between self-injective algebras.

Self-injective algebras

Let A be a finite dimensional self-injective k-algebra.

- Known: The stable category $\underline{\mathrm{mod}}$-A is a triangulated category and a natural quotient of the derived category $D^b(\mathrm{mod}\text{-}A)$.

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- Therefore, a derived equivalence between self-injective algebras induces a stable equivalence.
  - Very few examples of stable equivalences between self-injective algebras which are not induced by derived equivalences are known (Broué, 1994; Linckelmann, 1996).
  - For self-injective algebras of finite representation type, almost all stable equivalences are induced by derived equivalences (Asashiba, 2003; Dugas, 2013; Chan-Koenig-Liu, 2015; Li-Liu, 2023).

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  - Although the derived equivalences preserve tensor products and trivial extensions, this is not true for stable equivalences (even of Morita type) (Liu-Zhou-Zimmermann, 2017; Bouc-Zimmermann, 2017).
- **In order to understand the difference between derived equivalences and stable equivalences for self-injective algebras, it is important to construct more examples of stable equivalences between self-injective algebras that are usually not lifted to derived equivalences**.

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Dugas' construction

Recently in this direction Dugas gave two methods to construct stable auto-equivalences for local symmetric algebras (Dugas, J.Algebra, 2016), which are modeled after

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which yield auto-equivalences of the derived category of coherent sheaves on a variety.

- It is interesting that such stable auto-equivalences are in general not induced by auto-equivalences of the derived category.

Spherical stable twist

Theorem 1 (Dugas, 2016)

Let A be a elementary local symmetric k-algebra, which is free as both a left and a right module over a subalgebra $R = k[x] \cong k[t]/(t^m)$ $(m \geq 2)$. Assume that $\underline{\mathit{End}}_A(k \otimes_R A) \cong k[\psi]/(\psi^2)$ with ψ induced by left multiplication by an element $y \in A$. Let K be the kernel of the multiplication map $\mu : A \otimes_R A \to A$, then $- \otimes_A K$ induces an auto-equivalence of $\underline{\mathrm{mod}}$-A.

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Remark

Note that $Cone(\mu) = \Omega_{A^e}^{-1}(K)$ in $\underline{\mathrm{mod}}$-A^e, and the stable auto-equivalence $- \otimes_A \Omega_{A^e}^{-1}(K) : \underline{\mathrm{mod}}$-$A \to \underline{\mathrm{mod}}$-$A$ is called a **spherical stable twist**.

Theorem 2 (Dugas, 2016)

Let A be a elementary local symmetric k-algebra, which is free as both a left and a right module over a subalgebra $R = k[x] \cong k[t]/(t^m)$ $(m \geq 2)$. Assume that $\underline{End}_A(k \otimes_R A) \cong k[\psi]/(\psi^{n+1})$ for some $n \geq 1$, where ψ is induced by left multiplication by some $y \in A$ such that $xy = yx$. If we set

$$Q \cong Cone(Cone(A \otimes_R A \xrightarrow{y\otimes 1 - 1\otimes y} A \otimes_R A) \xrightarrow{\overline{\mu}} A)$$

in $\underline{\mathrm{mod}}$-A, then $- \otimes_A Q$ induces an auto-equivalence of $\underline{\mathrm{mod}}$-A.

The rough idea of the Proof

- Consider the **strong spanning class** $\mathcal{C} := \{T\} \cup T^{\perp}$ in the stable category $\underline{\mathrm{mod}}$-A (that is, $\mathcal{C}^{\perp} = \{0\}$ and ${}^{\perp}\mathcal{C} = \{0\}$), where $T := k \otimes_R A \cong A/(\mathrm{rad} R)A$.

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- (Bridgeland, 1999; Dugas, 2016) Let ${}_AM_A$ be a left-right projective A^e-module. Then $- \otimes_A M : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}A$ is an equivalence if and only if $- \otimes_A M$ induces bijections

$$\underline{\mathrm{Hom}}_A(X, \Omega^{-i}(Y)) \to \underline{\mathrm{Hom}}_A(X \otimes_A M, \Omega^{-i}(Y) \otimes_A M)$$

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for all $X, Y \in \mathcal{C}$ and for all $i = 0, 1$ (enough injections for $i = 1$).
- Restricted to $\mathrm{add}(T \oplus T[-1])$, the stable auto-equivalence is isomorphic to the identity functor on $\underline{\mathrm{mod}}\text{-}A$ (up to the auto-equivalence [1] or [2]).

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Our ideas

We wish to generalize Dugas' construction in the following respects.

- local $\to$ non-local
- a pair (A, R) $\to$ a triple (A, R, B)
- cone, double cone construction $\to$ multiple cone construction

Construction

Assumption 1: Let k be a field, A be a symmetric k-algebra, R be a non-semisimple symmetric k-subalgebra of A such that A_R is projective. Let B be another k-subalgebra of A, such that the following conditions hold:

(a) $br = rb$ for each $b \in B$ and $r \in R$;

(b) $B \otimes_k (R/radR) \xrightarrow{\phi} (R/radR) \otimes_R A$, $b \otimes \overline{1} \mapsto \overline{1} \otimes b$ is an isomorphism in $\underline{\mathrm{mod}}$-R;

(c) B has a periodic free B^e-resolution (of period q), that is, there exists an exact sequence

$$0 \to B \xrightarrow{\delta_q} (B \otimes_k B)^{m_{q-1}} \xrightarrow{\delta_{q-1}} \cdots \to (B \otimes_k B)^{m_1} \xrightarrow{\delta_1} (B \otimes_k B)^{m_0} \xrightarrow{\delta_0} B \to 0$$

of B^e-modules.

Construction

Under Assumption 1, there exists a complex

$$(A \otimes_R A)^{m_{q-1}} \xrightarrow{d_{q-1}} \cdots \to (A \otimes_R A)^{m_1} \xrightarrow{d_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A \to 0$$

in $\mathrm{lrp}(A)$ such that the diagram

$$\begin{array}{ccccccccc}
(B \otimes_k B)^{m_{q-1}} & \xrightarrow{\delta_{q-1}} \cdots \longrightarrow & (B \otimes_k B)^{m_1} & \xrightarrow{\delta_1} & (B \otimes_k B)^{m_0} & \xrightarrow{\delta_0} & B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
(A \otimes_R A)^{m_{q-1}} & \xrightarrow{d_{q-1}} \cdots \longrightarrow & (A \otimes_R A)^{m_1} & \xrightarrow{d_1} & (A \otimes_R A)^{m_0} & \xrightarrow{d_0} & A & \longrightarrow & 0
\end{array}$$

is commutative, where the vertical morphisms are the obvious morphisms.

Construction

We can factor out the complex

$$(A \otimes_R A)^{m_{q-1}} \xrightarrow{d_{q-1}} \cdots \to (A \otimes_R A)^{m_1} \xrightarrow{d_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A$$

into triangles

$$M_1 \xrightarrow{i_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A \to,$$

$$M_2 \xrightarrow{i_2} (A \otimes_R A)^{m_1} \xrightarrow{f_1} M_1 \to,$$

$$\cdots,$$

$$M_q \xrightarrow{i_q} (A \otimes_R A)^{m_{q-1}} \xrightarrow{f_{q-1}} M_{q-1} \to$$

in the triangulated category $\underline{\mathrm{lrp}}(A)$ such that $i_p f_p = d_p$ in $\mathrm{lrp}(A)$ for $1 \leq p \leq q-1$.

Construction

Theorem 3 (Li-Liu, 2023)

Let (A, R, B) be a triple which satisfies Assumption 1. If M_q is the A-A-bimodule defined as above, then $- \otimes_A M_q : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}A$ is a stable auto-equivalence of A.

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- In Dugas's construction, $T_A = A/(\mathrm{rad} R)A)$ has Ω_A-period 2, but in our construction, T_A may not be Ω_A-periodic. So our construction is more flexible.

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- The subalgebra B can be seen as a generalization of the algebra $\underline{\mathrm{End}}_A(A/(\mathrm{rad} R)A)$ in Dugas' construction. In fact, when R is elementary and local, B is Morita equivalent to $\underline{\mathrm{End}}_A(A/(\mathrm{rad} R)A)$.

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- Restricted to $\mathrm{add}(T \oplus T[-1])$, $-\otimes_A M_q$ is isomorphic to the identity functor on $\underline{\mathrm{mod}}\text{-}A$.

Example 1: Recover the endo-trivial modules over a group algebra of a finite p-group

Let k be a field of positive characteristic p, P be a **finite p-group** and kP be its **group algebra**.

- A kP-module M is called **endo-trivial** if $\mathrm{End}_k(M) \cong k \oplus P$ for some projective module P.
- Two endo-trivial modules M, N are said to be equivalent if $M \oplus Q_1 \cong N \oplus Q_2$ for some projective kP-modules Q_1, Q_2.
- **The group** $T(P)$ has elements consisting of equivalence classes $[M]$ of endo-trivial modules M. The operation is given by $[M] + [N] = [M \otimes_k N]$.

Example 1: Recover the endo-trivial modules over a group algebra of a finite p-group

Let $A = kP$ and $R = kS$, $B = kL$ for some subgroups S, L of P. Suppose that the triple (A, R, B) satisfies Assumption 1, and let $\rho_{S,L} := - \otimes_A M_q : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}A$ be the stable auto-equivalence of A in Theorem 3.

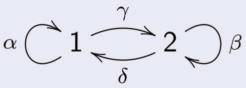
Example 1: Recover the endo-trivial modules over a group algebra of a finite p-group

Proposition (Li-Liu, 2023)

Let P be a finite p-group which is not generalized quaternion. Then there exist finitely many pairs $(S_i,\ L_i)$ of subgroups of P such that the following conditions hold:
(1) Each pair $(S_i,\ L_i)$ gives a triple (kP, kS_i, kL_i) which satisfies Assumption 1;
(2) $T(P)$ **is generated by** $[\Omega_{kP}(k)]$ **and elements of the form** $[\rho_{S_i,L_i}(k)]$.

Example 2

Let A be the symmetric k-algebra given by the quiver



with relations $(\alpha\delta\beta\gamma)^n = (\delta\beta\gamma\alpha)^n$, $(\beta\gamma\alpha\delta)^n = (\gamma\alpha\delta\beta)^n$, $\alpha^2 = \delta\gamma = \beta^2 = \gamma\delta = 0$. Let $R = k[\alpha] \times k[\beta]$, $B = k[x]$ be two subalgebras of A, where $x = (\delta\beta\gamma\alpha)^{n-1}\delta\beta\gamma + (\gamma\alpha\delta\beta)^{n-1}\gamma\alpha\delta$. The triple (A, R, B) satisfies Assumption 1.

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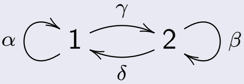

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Remark

A is an example of a **Brauer graph algebra** (that is, a symmetric special biserial algebra).

Example 2

When $n = 2$, the indecomposable projective A-modules have the following forms:

$$
e_1A = \begin{array}{ccc} & 1 & \\ 1 & & 2 \\ 2 & & 2 \\ 2 & & 1 \\ 1 & & 1 \\ 1 & & 2 \\ 2 & & 2 \\ 2 & & 1 \\ & 1 & \end{array}, \qquad
e_2A = \begin{array}{ccc} & 2 & \\ 2 & & 1 \\ 1 & & 1 \\ 1 & & 2 \\ 2 & & 2 \\ 2 & & 1 \\ 1 & & 1 \\ 1 & & 2 \\ & 2 & \end{array}.
$$

Example 2

(2.1) If $char(k) = 2$, then B has a periodic free B^e-resolution $0 \to B \to B \otimes_k B \xrightarrow{\mu} B \to 0$ **of period** 1, where μ is the map given by multiplication. The functor $- \otimes_A K$ induces a stable auto-equivalence of A, where K is the kernel of the A^e-homomorphism $A \otimes_R A \to A$ given by multiplication.

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Remark

When $n = 2$, it can be shown that the above auto-equivalence $- \otimes_A K$ **cannot be lifted to a derived auto-equivalence**, based on constructions of stable equivalences of Morita type (Liu-Xi, 2007) and constructions of derived equivalences (Hu-Xi, 2010).

Example 2

(2.2) If $char(k) \neq 2$, then B has a periodic free B^e-resolution $0 \to B \to B \otimes_k B \xrightarrow{f} B \otimes_k B \xrightarrow{\mu} B \to 0$ **of period** 2, where $f(1 \otimes 1) = 1 \otimes x - x \otimes 1$ and μ is the map given by multiplication. The functor $- \otimes_A K'$ induces a stable auto-equivalence of A, where K' is given by the short exact sequences $0 \to K' \to (A \otimes_R A) \oplus P \xrightarrow{(h_1, h_2)} K \to 0$ and $0 \to K \to A \otimes_R A \xrightarrow{m} A \to 0$ of A^e-modules. Here m is given by multiplication, $h_1(1 \otimes 1) = 1 \otimes x - x \otimes 1$, and $h_2 : P \to K$ is the projective cover of K as an A^e-module.

Example 2

Remark

If k is a splitting field for A, then all the stable auto-equivalences of A constructed above are indeed **stable auto-equivalences of Morita type**.

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Thank you!