Relative cluster tilting theory and τ-tilting theory

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Contents

Background

Tilting theory plays a important role in the representation theory of algebra. It has now evolved into an indispensable tool across various mathematical domains. Applications have been found in diverse fields such as finite and algebraic group theory, commutative and non-commutative algebraic geometry, as well as algebraic topology.

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Adachi, Iyama and Reiten introduced τ-*tilting theory*, which is a generalization of classical tilting theory. The impetus to explore τ-tilting theory arises from various sources, with a key focus on the mutation of tilting modules.

τ-tilting theory (categorical version)

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Definition (Iyama-Jørgensen-Yang, 2014)

Let $\mathcal{R}$ be an additive category.

(i) Let $\mathcal{M}$ be a subcategory of mod$\mathcal{R}$. A class

$$\{ P_1 \xrightarrow{\pi^M} P_0 \to M \to 0 \mid M \in \mathcal{M} \}$$

of projective presentations in mod$\mathcal{R}$ is said to have Property (S) if

$$\mathsf{Hom}_{\mathrm{mod}\mathcal{R}}(\pi^M, M') \colon \mathsf{Hom}_{\mathrm{mod}\mathcal{R}}(P_0, M') \to \mathsf{Hom}_{\mathrm{mod}\mathcal{R}}(P_1, M')$$

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is surjective for any $M, M' \in \mathcal{M}$.

(ii) A subcategory $\mathcal{M}$ of mod$\mathcal{R}$ is said to be τ-rigid if there is a class of projective presentations $\{P_1 \to P_0 \to M \to 0 \mid M \in \mathcal{M}\}$ which has Property (S).

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Definition (Iyama-Jørgensen-Yang, 2014)

Let $\mathcal{R}$ be an additive category.

(iii) A τ-rigid pair of mod$\mathcal{R}$ is a pair $(\mathcal{M}, \mathcal{E})$, where $\mathcal{M}$ is a τ-rigid subcategory of mod$\mathcal{R}$ and $\mathcal{E} \subseteq \mathcal{R}$ is a subcategory with $\mathcal{M}(\mathcal{E}) = 0$, that is, $M(E) = 0$ for each $M \in \mathcal{M}$ and $E \in \mathcal{E}$.

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(iv) A τ-rigid pair $(\mathcal{M}, \mathcal{E})$ is support τ-tilting if $\mathcal{E} = \mathrm{Ker}\,(\mathcal{M})$ and for each $R \in \mathcal{R}$ there exists an exact sequence

$$\mathcal{R}(-, R) \xrightarrow{f} M^0 \to M^1 \to 0$$

with $M^0, M^1 \in \mathcal{M}$ such that f is a left $\mathcal{M}$-approximation.
In this case, $\mathcal{M}$ is called a support τ-tilting subcategory of mod$\mathcal{R}$.

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O. Iyama, P. Jørgensen and D. Yang. Intermediate co-t-structures, two-term silting objects, τ-tilting modules, and torsion classes. Algebra and Number Theory, 8(10), 2413-2431, 2014.

Support τ-tilting modules and cluster tilting objects

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Theorem (Buan-Marsh-Reiten, 2007 and Keller-Reiten, 2007)

Let R be a cluster tilting object in $\mathscr{C}$ with endomorphism algebra $\Lambda = \mathsf{End}_{\mathscr{C}}(R)$. Then the functor

$$\mathbb{H} := \mathrm{Hom}_{\mathscr{C}}(R, -) \colon \mathscr{C} \longrightarrow \mathrm{mod}\Lambda$$

induces an equivalence

$$\mathscr{C}/\mathrm{add}(R[1]) \xrightarrow{\simeq} \mathrm{mod}\Lambda.$$

A. B. Buan, R. Marsh and I. Reiten. Cluster-tilted algebras. Trans. Amer. Math. Soc. 359, 323-332, 2007.

B. Keller and I. Reiten. Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. Adv. Math. 211, 123-151, 2007.

Support τ-tilting modules and cluster tilting objects

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Theorem (Adachi-Iyama-Reiten, 2014)

Let $\mathscr{C}$ be a 2-Calabi-Yau triangulated category with a cluster tilting object R, and $\Lambda = \mathsf{End}_{\mathscr{C}}(R)$. Then the functor $\mathbb{H} := \mathsf{Hom}_{\mathscr{C}}(R, -)$ induces the following bijections

1. *Rigid objects in $\mathscr{C}$ $\stackrel{1-1}{\longleftrightarrow}$ τ-rigid pairs in $\mathrm{mod}\Lambda$.*
2. *Cluster tilting objects in $\mathscr{C}$ $\stackrel{1-1}{\longleftrightarrow}$ Support τ-tilting pairs in $\mathrm{mod}\Lambda$.*

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Unfortunately, many examples indicate that this result does not hold if $\mathscr{C}$ is not 2-Calabi-Yau. It is then reasonable to find a class of objects in arbitrary triangulated categories $\mathscr{C}$ which correspond to support τ-tilting modules in $\operatorname{mod}\Lambda$ bijectively in more general setting.

Relative rigid subcategories and related subcategories

Let $\mathscr{C}$ be a triangulated category and $\mathcal{R}$ a rigid subcategory of $\mathscr{C}$. Then the functor

$$\mathbb{H} := \mathrm{Hom}_{\mathscr{C}}(\mathcal{R}, -) \colon \mathcal{R} * \mathcal{R}[1] \longrightarrow \mathrm{Mod}\mathcal{R}$$

induces an equivalence

$$(\mathcal{R} * \mathcal{R}[1])/\mathcal{R}[1] \xrightarrow{\simeq} \mathrm{mod}\mathcal{R}.$$

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It is natural to ask which class of subcategories in $\mathcal{R} * \mathcal{R}[1]$ correspond to support τ-tilting subcategories of $\mathrm{mod}\mathcal{R}$ bijectively.

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Motivated by this question, we introduce the notion of relative rigid subcategories.

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Definition

Let $\mathscr{C}$ be a triangulated category and $\mathcal{R}$ a rigid subcategory of $\mathscr{C}$.

(i) A subcategory $\mathscr{X}$ in $\mathscr{C}$ is called $\mathcal{R}[1]$-rigid if $[\mathcal{R}[1]](\mathscr{X}, \mathscr{X}[1]) = 0$.

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(ii) A subcategory $\mathscr{X} \subseteq \mathcal{R}*\mathcal{R}[1]$ is called two-term weak $\mathcal{R}[1]$-cluster tilting if $\mathcal{R} \subseteq \mathscr{X}[-1] * \mathscr{X}$ and

$$\mathscr{X} = \{M \in \mathcal{R}*\mathcal{R}[1] \mid [\mathcal{R}[1]](M, \mathscr{X}[1]) = 0 \text{ and } [\mathcal{R}[1]](\mathscr{X}, M[1]) = 0\}.$$

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(iii) An object X is called two-term $\mathcal{R}[1]$-rigid, two-term weak $\mathcal{R}[1]$-cluster tilting if $\mathrm{add}X$ is two-term $\mathcal{R}[1]$-rigid, two-term weak $\mathcal{R}[1]$-cluster tilting respectively.

Example

Example (Relative cluster tilting is not always cluster tilting)

Let $A = kQ/I$ be a self-injective algebra given by the quiver

$$Q: \; 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

and $I = \langle \alpha\beta\alpha\beta, \beta\alpha\beta\alpha \rangle$.

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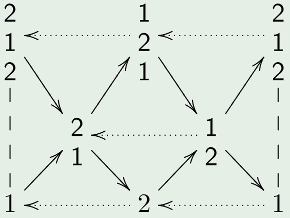

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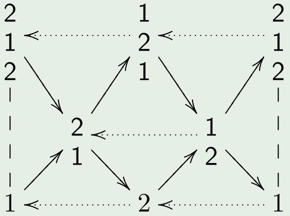

$\underline{\mathcal{R}} := \mathsf{add}(2 \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix})$

$\underline{\mathscr{X}} := \mathsf{add}(2 \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix})$ is a weak $\underline{\mathcal{R}}[1]$-cluster tilting subcategory, but not a cluster tilting subcategory.

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Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. Then

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(2) *$\mathcal{X}$ is two-term weak $\mathcal{R}[1]$-cluster tilting if and only if $\overline{\mathcal{X}}$ is a support τ-tilting subcategory.*

Bongartz and co-Bongartz completions

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Proposition

If each object in $\mathcal{R}$ admits a left $\mathcal{X}$-approximation. Then $\mathcal{X}$ is contained in a two-term weak $\mathcal{R}[1]$-cluster tilting subcategory $\mathcal{M}_{\mathcal{X}}$:

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Proposition (Zhou-Zhu, 2020)

If each object in $\mathcal{R}[1]$ admits a right $\mathcal{X}$-approximation. Then $\mathcal{X}$ is contained in a two-term weak $\mathcal{R}[1]$-cluster tilting subcategory $\mathcal{N}_{\mathcal{X}}$:

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$\mathcal{N}_{\mathcal{X}}$ (resp. $\mathcal{M}_{\mathcal{X}}$) is called the (co-)Bongartz completion of $\mathcal{X}$.

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Theorem

Let $\mathcal{X}$ be an $\mathcal{R}[1]$-functorially finite two-term $\mathcal{R}[1]$-rigid subcategory. Then $(\mathcal{M}_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}})$ is an $\mathcal{X}$-mutation pair.

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Proposition

Let $\mathcal{U}$ be a two-term $\mathcal{R}[1]$-rigid subcategory. If $(\mathcal{M}, \mathcal{N})$ is a $\mathcal{U}$-mutation pair, then $\mathcal{M} = \mathcal{M}_{\mathcal{U}}$ and $\mathcal{N} = \mathcal{N}_{\mathcal{U}}$. Moreover, $\mathcal{U}$ is $\mathcal{R}[1]$-functorially finite.

Completions of almost complete subcategories

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Theorem

Let $\mathcal{X}$ be an almost complete two-term weak $\mathcal{R}[1]$-cluster tilting subcategory. Then $\mathcal{M}_{\mathcal{X}}$ and $\mathcal{N}_{\mathcal{X}}$ are completions of $\mathcal{X}$. Moreover, if $\mathcal{L}$ is a two-term weak $\mathcal{R}[1]$-cluster tilting subcategory which contains $\mathcal{X}$, then $\mathcal{L} = \mathcal{M}_{\mathcal{X}}$ or $\mathcal{L} = \mathcal{N}_{\mathcal{X}}$.

τ-tilting theory in abelian categories

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For any subcategory $\overline{\mathcal{U}} \subseteq \overline{\mathcal{A}}$, denote

$$\{A \in \overline{\mathcal{A}} \mid \mathrm{Ext}^1_{\mathcal{A}}(\overline{\mathcal{U}}, \mathrm{Fac}(\mathrm{add}A)) = 0\} \text{ by } {}^{\perp}(\tau\overline{\mathcal{U}}).$$

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$$\mathrm{Fac}\overline{\mathcal{M}}_{\mathcal{X}} = \mathrm{Fac}\overline{\mathcal{X}},\ \mathrm{Fac}\overline{\mathcal{N}}_{\mathcal{X}} = {}^{\perp}(\tau\overline{\mathcal{X}}) \cap \mathcal{E}^{\perp} \textit{ and } \mathrm{Fac}\overline{\mathcal{M}}_{\mathcal{X}} \subsetneq \mathrm{Fac}\overline{\mathcal{N}}_{\mathcal{X}}.$$

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Let R be a commutative noetherian ring which is complete and local. Let $\widehat{\mathcal{A}}$ be an Ext-finite abelian category over R with enough projectives. Let $\mathcal{P}$ be the subcategory of projective objects.

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Let $\mathcal{C} = \mathrm{D}^b(\widehat{\mathcal{A}})$. Then $\mathcal{C}$ is Krull-Schmidt. Moreover, it is Hom-finite over R. Here we denote $\mathcal{P} * \mathcal{P}[1]$ by $\mathcal{A}$ and $(\mathcal{P} * \mathcal{P}[1])/\mathcal{P}[1]$ by $\overline{\mathcal{A}}$.

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Proposition

We have an equivalence of additive categories: $\overline{\mathcal{A}} \simeq \widehat{\mathcal{A}}$.

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(X0) $\widehat{\mathcal{X}}$ *is not support τ-tilting;*

(X1) $\widehat{\mathcal{X}}$ *is contravariantly finite;*

(X2) *every projective object admits a left $\widehat{\mathcal{X}}$-approximation;*

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Then it is contained in two support τ-tilting pairs $(\mathcal{M}, \mathcal{E})$ and $(\mathcal{N}, \mathcal{E})$ such that

$$\mathrm{Fac}\mathcal{M} = \mathrm{Fac}\widehat{\mathcal{X}},\ \mathrm{Fac}\mathcal{N} = {}^{\perp}(\tau\widehat{\mathcal{X}}) \cap \mathcal{E}^{\perp} \text{ and } \mathrm{Fac}\mathcal{M} \subsetneq \mathrm{Fac}\mathcal{N}.$$

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Moreover, if $(\widehat{\mathcal{X}}, \mathcal{E})$ is an almost complete support τ-tilting pair, then $(\mathcal{M}, \mathcal{E})$ and $(\mathcal{N}, \mathcal{E})$ are the only support τ-tilting pairs which contain $(\widehat{\mathcal{X}}, \mathcal{E})$.

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We introduce the following notions:

(1) $\widehat{\mathcal{A}}_{c\text{-}s\tau\text{-}til}$ =: { contravariantly finite support τ-tilting subcategories in $\widehat{\mathcal{A}}$ };

(2) $\widehat{\mathcal{A}}_{lw\text{-}ctp}$ =: { left weak cotorsion torsion pairs in $\widehat{\mathcal{A}}$ };

(3) $\widehat{\mathcal{A}}_{\tau\text{-}ctp}$ =: { τ-cotorsion torsion pairs in $\widehat{\mathcal{A}}$ };

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Theorem

There are bijections

$$\widehat{\mathcal{A}}_{c\text{-}s\tau\text{-}til} \longleftrightarrow \widehat{\mathcal{A}}_{f\text{-}tor} \longleftrightarrow \widehat{\mathcal{A}}_{lw\text{-}ctp} = \widehat{\mathcal{A}}_{\tau\text{-}ctp}$$

given by

(a1) $\widehat{\mathcal{A}}_{c\text{-}s\tau\text{-}til} \ni \mathcal{M} \mapsto \mathrm{Fac}\mathcal{M} \in \widehat{\mathcal{A}}_{f\text{-}tor}$;
(a2) $\widehat{\mathcal{A}}_{f\text{-}tor} \ni \mathcal{T} \mapsto {}^{\perp_1}\mathcal{T} \cap \mathcal{T} \in \widehat{\mathcal{A}}_{c\text{-}s\tau\text{-}til}$;
(b1) $\widehat{\mathcal{A}}_{f\text{-}tor} \ni \mathcal{T} \mapsto ({}^{\perp_1}\mathcal{T}, \mathcal{T}) \in \widehat{\mathcal{A}}_{lw\text{-}ctp}$;
(b2) $\widehat{\mathcal{A}}_{lw\text{-}ctp} \ni (\mathcal{S}, \mathcal{T}) \mapsto \mathcal{T} \in \widehat{\mathcal{A}}_{f\text{-}tor}$.

Thank you!