Categorical properties and homological conjectures for bounded extensions of algebras

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Aug 8, 2024

Outline

Notations

- A: f.d algebra over a field k.
- $\mathrm{Mod}A$: category of left A-modules.
- $\mathrm{mod}A$: category of finitely generated left A-modules.
- DA: derived category of complexes over $\mathrm{Mod}A$.
- $\mathcal{D}^b(\mathrm{mod}A)$: bounded derived category of complexes over $\mathrm{mod}A$.

Singularity category

- per$A$: full subcategory of $D(A)$ consisting of all compact complexes.
- proj$A$: category of finitely generated projective A-modules.
- $K^b(\text{proj}A)$: bounded homotopy category of complexes over proj$A$.
- Up to iso, $\text{per}A = K^b(\text{proj}A)$.
- $D_{sg}(A) = \mathcal{D}^b(\text{mod}A)/K^b(\text{proj}A)$: singularity category.
- Fact: $D_{sg}(A) = 0$ iff $\text{gl.dim}A < \infty$.

Singular equivalence

A and B are singularly equivalent if $D_{sg}(A) \simeq D_{sg}(B)$ as tri cat.

Definition [Wang 15]

Let M be an A-B-bimodule, N be a B-A-bimodule and $l \in \mathbb{N}$. $({}_AM_B, {}_BN_A)$ defines a singular equivalence of Morita type with level l if ${}_AM$, M_B, ${}_BN$ and N_A are projective, and

$$M \otimes_B N \cong \Omega^l_{A^e}(A) \text{ and } N \otimes_A M \cong \Omega^l_{B^e}(B).$$

In this case, there is a triangle equivalence

$$M \otimes_B - : D_{sg}(B) \to D_{sg}(A).$$

Many homological conjectures and homological properties are invariant under singular equivalence of Morita type with level, such as:

- the finitistic dimension conjecture
- Keller's conjecture for singular Hochschild cohomology
- the properties of syzygy-finite and Igusa-Todorov
- injectives generation

So it is interesting to construct singular equivalences of Morita type with level.

Ref:[Dalezios 2021][Chen-Liu-Wang 2023] [Qin 2022]

Gorenstein projective modules

$M \in \mathrm{mod}A$ is called Gorenstein projective if there is an exact sequence

$$P^{\bullet} = \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^{0} \xrightarrow{d^{0}} P^{1} \longrightarrow \cdots$$

of $\mathrm{proj}A$ with $M = \mathrm{Ker}d^0$ such that $\mathrm{Hom}_A(P^{\bullet}, Q)$ is exact for every $Q \in \mathrm{proj}A$.

- $\mathrm{Gproj}A$: the category of Gorenstein projective modules.
- $\underline{\mathrm{Gproj}}A$: the stable category of Gorenstein projective modules.

Buchweitz's Theorem

$\mathrm{Gproj}A$ is a Frobenius category, and there is a tri. embedding

$$F : \underline{\mathrm{Gproj}}A \hookrightarrow D_{sg}(A).$$

Definition [Bergh-Jørgensen-Oppermann 2015]

The Verdier quotient $D_{def}(A) := D_{sg}(A)/\mathrm{Im}F$ is called the Gorenstein defect category of A.

Fact: A is Gorenstein iff $D_{def}(A) = 0$.

Comparison

Consider the exact sequence of triangulated categories:

$$0 \longrightarrow \underline{\mathrm{Gproj}}A \longrightarrow D_{sg}(A) \longrightarrow D_{def}(A) \longrightarrow 0$$

Question

Let A and B be two related algebras. When

$\underline{\mathrm{Gproj}}A \simeq \underline{\mathrm{Gproj}}B$? $D_{sg}(A) \simeq D_{sg}(B)$? $D_{def}(A) \simeq D_{def}(B)$?

Question

Question ($\cdots$ continued) When

$\underline{\mathrm{Gproj}} A \simeq \underline{\mathrm{Gproj}} B$? $D_{sg}(A) \simeq D_{sg}(B)$? $D_{def}(A) \simeq D_{def}(B)$?

- [Chen 2009][PSS 2014][Shen 2021][LHZ 2022][QS 2024]: A and eAe.
- [Chen 2014][ZLM 2024]: A and A/I.
- [Lu 2019]: simple gluing algebras.
- [EPS 2022]: $A = kQ/I$ and $A/\langle\alpha\rangle$.

In this talk, I will compare these categories and homological conjectures between A and B when $B \subset A$ is a ring extension.

Definition [Cibils-Lanzilotta-Marcos-Solotar 2022]

An extension $B \subset A$ of finite dimensional algebras is called left (resp. right) bounded if

(a) $(A/B)^{\otimes_B p} = 0$ for some $p \geq 1$;

(b) $\mathrm{pd}_{B^e}(A/B) < \infty$;

(c) ${}_B(A/B)$ (resp. $(A/B)_B$) is projective.

Definition

An extension $B \subset A$ is called bounded if both (a) and (b) hold, and (c) is replaced by

(c′): $\mathrm{Tor}_i^B(A/B, (A/B)^{\otimes_B j}) = 0$ for all $i, j \geq 1$.

$\Leftrightarrow \mathrm{Tor}_i^B((A/B)^{\otimes_B j}, A/B) = 0$ for all $i, j \geq 1$.

Examples of bounded extensions

- Let $A := B \ltimes M$ be the trivial extension. Then $B \subset A$ is bounded if $\mathrm{pd}({}_B M_B) < \infty$, $M^{\otimes_B p} = 0$ for some p and $\mathrm{Tor}_i^B(M, M^{\otimes_B j}) = 0$ for each $i, j \geq 1$;
- Arrow removal operation. $A = kQ/I$, α: an arrow not in the generating set of I, $B := A/\langle\overline{\alpha}\rangle$. Then $B \subset A$ is a bounded extension.
- Let $A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$ be a triangular matrix algebra. Then $B \times C \subset A$ is a bounded extension if $\mathrm{pd}_C M_B < \infty$.
- Morita context algebra with certain conditions.

Example

Let $Q: 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$, $B = kQ$, $S(i)$ (resp. $T(i)$):the simple left (resp. right) B-module corresponding to i. Let $M := S(3) \otimes_k T(2)$ and $A := B \ltimes M$. Then $B \subset A$ is a bounded extension, but not left or right bounded in the sense of Cibils et. al.

Compare

{left (resp. right) bounded extensions} $\subsetneqq$ {bounded extensions}

Theorem I [Qin-Xu-Zhang-Zhou 2024]

Let $B \subset A$ be a bounded extension. Then

$$_AA \otimes^{\mathbb{L}}_B - : D_{sg}(B) \rightleftarrows D_{sg}(A) :_B A \otimes_A -$$

is an equivalence which induces a singular equivalence of Morita type with level between A and B, and the functor $_BA \otimes_A - : \mathrm{mod}(A) \to \mathrm{mod}(B)$ is an eventually homological isomorphism.

$F : \mathrm{mod}(A) \to \mathrm{mod}(B)$ is an eventually homological isomorphism if $\exists\, t$ such that $\mathrm{Ext}^j_A(X, Y) \cong \mathrm{Ext}^j_B(FX, FY)$, for any $j > t$ and any $X, Y \in \mathrm{mod}(A)$. [PSS 2014]

Theorem I ($\cdots$continued)

Let $B \subset A$ be a bounded extension. If one of the following two conditions holds:

(1) The extension $B \subset A$ is split.

(2) $\mathbb{R}\mathrm{Hom}_B(A, B)$ is quasi-isomorphic to a bounded complex with each term in $\mathrm{Gproj}(A)^{\perp}$;

then there are triangle equivalences

$$\underline{\mathrm{Gproj}}A \simeq \underline{\mathrm{Gproj}}B \text{ and } D_{def}(A) \simeq D_{def}(B)$$

Here, $\mathrm{Gproj}(A)^{\perp} := \{X \in \mathrm{mod}A \mid \mathrm{Ext}^i_A(U, X) = 0 \text{ for all } U \in \mathrm{Gproj}(A), i \geq 1\}$.

Sketch of the proof

adjoint triple $DA \underset{\longleftarrow}{\overset{F=A\otimes^{\mathbb{L}}_B -}{\longleftarrow \atop \xrightarrow{G={}_BA\otimes_A -}}} DB$

- When F and G restrict to D_{sg}? equivalent?
  [OPS 2019] + [Dalezios 21] + normalised relative bar resolution [CLMS 2021]

- When F and G restrict to Gproj and D_{def}?
  [HP 2017] + [OPS 2019] + two criteria for a functor ${}_AX \otimes^{\mathbb{L}}_B -$ restrict to D_{def}.

- Finitistic dimension conjecture (FDC): $\mathrm{Sup}\{\ \mathrm{pd}M \mid \mathrm{pd}M < \infty\ \} < \infty$.
- Auslander-Reiten Conjecture (ARC): $\mathrm{Ext}^i_A(M, M \oplus A) = 0$ for all $i > 0 \Rightarrow M \in \mathrm{proj}A$.
- Keller's conjecture (KC): the singular Hochschild cohomology complex $C^*_{sg}(A, A)$ iso to the Hochschild cochain complex $C^*(S_{dg}(A), S_{dg}(A))$, where $S_{dg}(A)$ is the dg enhancement of $D_{sg}(A)$.

Homological conjectures

- Han's conjecture (HC):
  $HH_n(A) = 0$ for $n >> 0 \Rightarrow \text{gl.dim}(A) < \infty$.
- Fg condition:
  $HH^*(A)$ is Noetherian and $\text{Ext}^*_A(A/\text{rad}A, A/\text{rad}A)$ is a f. g $HH^*(A)$-module.

Theorem II [Qin-Xu-Zhang-Zhou 2024]

Let $B \subset A$ be a bounded extension. Then the following statements hold:

(1) B satisfies FDC if and only if so does A.

(2) B satisfies HC if and only if so does A.

(3) B satisfies KC if and only if so does A.

(2) extends the main result of [CLMS 2022] form left bounded extensions to bounded extensions.

Theorem II ($\cdots$continued)

Let $B \subset A$ be a bounded extension. If $\mathbb{R}\mathrm{Hom}_B(A, B)$ is perfect as left A-module or $B \subset A$ is split, then

(4) B is Gorenstein if and only if so does A.

(5) B satisfies ARC if and only if so does A.

(6) B satisfies the Fg condition if and only if so does A.

Applications

Arrow removal operation

$A = kQ/I$, α: an arrow not in the generating set of I, $B := A/\langle\overline{\alpha}\rangle$. Then $B \subset A$ is a bounded split extension. Applying Theorem I, II, we reobtain the main result of [EPS 2022]:

- A Gor iff B Gor;
- A satisfies Fg iff so does B;
- $D_{sg}(A) \simeq D_{sg}(B)$;
- ${}_B A_A \otimes -$ is an eventually homological isomorphism

Triangular matrix algebra

Let B and C be f.d. algebras, and M be a f.g C-B bimodule. Let $A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$ be a triangular matrix algebra. If $\mathrm{pd}_C M_B < \infty$, then $B \times C \subset A$ is a bounded split extension. Applying Theorem I, we get

$$D_{sg}(A) \cong D_{sg}(B) \coprod D_{sg}(C),$$

$$\underline{\mathrm{Gproj}} A \cong \underline{\mathrm{Gproj}} B \coprod \underline{\mathrm{Gproj}} C,$$

$$D_{def}(A) \cong D_{def}(B) \coprod D_{def}(C).$$

This generalizes a result of [Lu 2017]: If $\mathrm{pd}_C M_B = 0$, then $\cdots$.

Morita context algebra

Let $A := \begin{pmatrix} \Lambda & {}_\Lambda V_\Gamma \\ {}_\Gamma W_\Lambda & \Gamma \end{pmatrix}_{(\phi,\psi)}$ be a Morita context algebra and $B := \Lambda \times \Gamma$. Suppose that

(1) $W \otimes_\Lambda V = 0$ or $V \otimes_\Gamma W = 0$;

(2) $\mathrm{pd}({}_\Gamma W_\Lambda) < \infty$ and $\mathrm{pd}({}_\Lambda V_\Gamma) < \infty$;

(3) $\mathrm{Tor}_i^\Lambda(W, V) = 0$ and $\mathrm{Tor}_i^\Gamma(V, W) = 0$ for $i \geq 1$.

Then $B \subset A$ is a bounded extension and $D_{sg}(A) \simeq D_{sg}(\Lambda) \coprod D_{sg}(\Gamma)$.

If, in addition, $W \otimes_\Lambda V = 0$ and $V \otimes_\Gamma W = 0$, then $B \subset A$ splits and $\mathrm{Gproj}(A) \cong \mathrm{Gproj}(\Lambda) \coprod \mathrm{Gproj}(\Gamma)$, $D_{def}(A) \cong D_{def}(\Lambda) \coprod D_{def}(\Gamma)$.

Example

$$A:\ \gamma \circlearrowright 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \qquad\qquad B:\ \gamma \circlearrowright 1 \xrightarrow{\alpha} 2$$

$$\gamma^2 = 0 = \beta\alpha \qquad\qquad \gamma^2 = 0$$

Then $B \subset A$ is a bounded split extension. Applying Theorem I, we have that $D_{sg}(A) \cong D_{sg}(B)$, $\underline{\text{Gproj}}A \cong \underline{\text{Gproj}}B$, $D_{def}(A) \cong D_{def}(B)$.

Note: We remove the arrow β which lies in the generating set of I, not the case of arrow removal operation in [EPS 22] and [GPS 21].

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Thank you!