Lusztig sheaves and tensor products of integrable highest weight modules

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Contents

Lusztig's sheaves

Given a symmetric Cartan datum $(I, (-,-))$, one can define the associated quantum group $\mathbf{U} = \mathbf{U}_v(\mathfrak{g})$ and its oriented quiver $Q = (I, H, \Omega)$.
C.M.Ringel [R90] has considered the Hall algebra (and its composition sub-algebra) of the category of $\mathbb{F}_q Q$-representations $\mathbf{rep}_{\mathbb{F}_q}(Q)$ and provided a realization of $\mathbf{U}^+_{v=\sqrt{q}}(\mathfrak{g})$, the specialization of the positive part of the quantum group.
Then inspired by Ringel, G.Lusztig[L90,91] has defined a sheaf theoretic Hall algebra by using perverse complexes on the moduli space of $\mathbf{k}Q$-representations and provided the construction of canonical bases.

Moduli space and Lusztig's sheaves

Let $\mathbf{k} = \overline{\mathbb{F}}_q$ be the algebraic closure of the finite field $\mathbb{F}_q$. Given $\nu \in \mathbb{N}[I]$ and an I-graded k-vector space $\mathbf{V}$ of dimension vector $|\mathbf{V}| = \nu$,

$$\mathbf{E}_{\mathbf{V},\Omega} = \bigoplus_{h \in \Omega} \mathbf{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}).$$

The algebraic group $G_{\mathbf{V}} = \prod_{i \in I} \mathbf{GL}(\mathbf{V}_i)$ acts on $\mathbf{E}_{\mathbf{V},\Omega}$ by

$$(g \cdot x)_h = g_{h''} x_h g_{h'}^{-1}.$$

Let $\mathcal{D}^b_{G_{\mathbf{V}}}(\mathbf{E}_{\mathbf{V},\Omega})$ be the $G_{\mathbf{V}}$-equivariant derived category of constructible l-adic sheaves on $\mathbf{E}_{\mathbf{V},\Omega}$. The category $\mathcal{D}^b_{G_{\mathbf{V}}}(\mathbf{E}_{\mathbf{V},\Omega})$ contains a full subcategory $\mathcal{Q}_{\mathbf{V}}$ consisting of semisimple perverse complexes, which is called the category of Lusztig's sheaves. The Verdier duality $\mathbf{D}$ acts on $\mathcal{Q}_{\mathbf{V}}$.

Induction and Restriction functors

Given $\nu' + \nu'' = \nu \in \mathbb{N}[I]$ and graded vector spaces $\mathbf{V}, \mathbf{V}', \mathbf{V}''$ of dimension vectors ν, ν', ν'' respectively, Lusztig's introduced induction functor and restrcition functor

$$\mathbf{Ind}^{\mathbf{V}}_{\mathbf{V}',\mathbf{V}''} : \mathcal{D}^b_{G_{\mathbf{V}'}}(\mathbf{E}_{\mathbf{V}',\Omega}) \times \mathcal{D}^b_{G_{\mathbf{V}''}}(\mathbf{E}_{\mathbf{V}'',\Omega}) \to \mathcal{D}^b_{G_{\mathbf{V}}}(\mathbf{E}_{\mathbf{V},\Omega}),$$

$$\mathbf{Res}^{\mathbf{V}}_{\mathbf{V}',\mathbf{V}''} : \mathcal{D}^b_{G_{\mathbf{V}}}(\mathbf{E}_{\mathbf{V},\Omega}) \to \mathcal{D}^b_{G_{\mathbf{V}'} \times G_{\mathbf{V}''}}(\mathbf{E}_{\mathbf{V}',\Omega} \times \mathbf{E}_{\mathbf{V}'',\Omega}).$$

Lusztig's main Theorem

Theorem (Lusztig,1990-1991)

With the induction and restriction functors, the Grothendieck group $\mathcal{K}$ of $\coprod_{\mathbf{V}} \mathcal{Q}_{\mathbf{V}}$ becomes a bialgebra, and is canonically isomorphic to the (integral form of) positive part of the quantized enveloping algebra ${}_{\mathcal{A}}\mathbf{U}^+$.

The images of simple perverse sheaves in $\mathcal{P}_{\mathbf{V}}$ form a $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$-basis of ${}_{\mathcal{A}}\mathbf{U}^+_{\mathbf{V}}$, which is called the canonical basis.

Properties of canonical basis

The canonical basis $\mathcal{B} = \{b = [L] | L \in \mathcal{P}\}$ has many remarkable properties:
(1) $\bar{b} = b$;
(2) $\mathcal{B}$ is a $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ basis of the integral form $\mathbf{f} \cong {}_{\mathcal{A}}\mathbf{U}^-$;
(3) $b * b' \in \sum\limits_{b'' \in \mathcal{B}} \mathbb{N}[v, v^{-1}] b''$ and similar for comultiplication;
(4) There exists a subset of $\mathcal{B}$ which forms a basis of the left ideal $\sum\limits_{i \in I} {}_{\mathcal{A}}\mathbf{U}^- f_i^{(d_i)}$.

Algebraic realization of $L(\Lambda)$

Given a dominant weight Λ, one can define the irreducible highest weight module $L(\Lambda)$ of $\mathbf{U}_v(\mathfrak{g})$. Consider the canonical map

$$\pi : {}_{\mathcal{A}}\mathbf{U}^- \to {}_{\mathcal{A}}\mathbf{U}^- / \sum_{i\in I} {}_{\mathcal{A}}\mathbf{U}^- f_i^{\langle \Lambda, \alpha_i^\vee \rangle + 1} \cong {}_{\mathcal{A}}L(\Lambda),$$

then due to the fourth property of the canonical basis, $\{\pi([L]) \neq 0 | L \in \mathcal{P}\}$ forms a basis of ${}_{\mathcal{A}}L(\Lambda)$, which is called the canonical basis of ${}_{\mathcal{A}}L(\Lambda)$.

Canonical basis of tensor products

Bao and Wang [BW16] generalized Lusztig's canonical basis $\{b \diamond b'\}$ (in [L92]) of $L(\Lambda_1) \otimes L^{\omega}(\Lambda_2)$ to canonical basis of $L(\Lambda_1) \otimes L(\Lambda_2)$ (and other more general cases). These bases are invariant under a certain involution and satisfies some positivity (proved by many other people)

$$E_i, F_i(b \diamond b') \in \sum \mathbb{N}[v, v^{-1}] b'' \diamond b''',$$

$$b \diamond b' \in b \otimes b' + \sum \mathbb{N}[v, v^{-1}] b'' \otimes b'''.$$

However, a geometric realization of tensor products and their canonical bases is still expected.

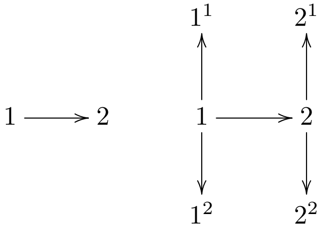
Motivation

Our goal is to provide a catgegorification of certain tensor products and their canonical bases based on perverse sheaves of quiver representations.

N-framed quivers and $\mathbf{E}_{\mathbf{V},\mathbf{W}^\bullet,\Omega^N}$

Definition

For the quiver $Q = (I, H, \Omega)$ and a given positive integer N, define the N-framed quiver of Q to be the quiver $Q^{(N)} = (I^{(N)}, H^{(N)}, \Omega^N)$, where the set of vertices $I^{(N)} = I \cup \{i^k | i \in I, 1 \leqslant k \leqslant N\}$ contains $N+1$ copies of I, the set of edges is $H \cup \{i \to i^k, i^k \to i | i \in I, 1 \leqslant k \leqslant N\}$, and the orientation Ω^N is $\Omega \cup \{i \to i^k | i \in I, 1 \leqslant k \leqslant N\}$.



Given a sequence of dominating weights $\Lambda^\bullet = (\Lambda_1, \Lambda_2, \cdots \Lambda_N)$, we fix a sequence of graded spaces $\mathbf{W}^\bullet = (\mathbf{W}^1, \mathbf{W}^2, \cdots \mathbf{W}^N)$ such that each $\mathbf{W}^k$ is I^k-graded spaces and $\langle \Lambda_k, \alpha_i^\vee \rangle = \omega_i^k = \mathrm{dim} \mathbf{W}_{i^k}$. ($\mathbf{W}^k$ supported on k-th copy of I.)
Define the moduli space of N-framed quivers for $\mathbf{V} \oplus \mathbf{W}^\bullet$ by

$$\mathbf{E}_{\mathbf{V},\mathbf{W}^\bullet,\Omega^N} = \mathbf{E}_{\mathbf{V},\Omega} \oplus \bigoplus_{i \in I, 1 \leqslant k \leqslant N} \mathbf{Hom}(\mathbf{V}_i, \mathbf{W}_{i^k}).$$

Thick subcategory $\mathcal{N}_{\mathbf{V},i}$

Fix $i \in I$ and orientation Ω^i such that i is a source, the subsets

$$\mathbf{E}^r_{\mathbf{V},\mathbf{W}^\bullet,i} = \{x \in \mathbf{E}_{\mathbf{V},\mathbf{W}^\bullet,\Omega^{i,N}} |$$

$$\mathrm{dimKer}(\bigoplus_{h\in\Omega^{i,N},h'=i} x_h : \mathbf{V}_i \to \bigoplus_{1\leqslant k\leqslant N} \mathbf{W}^k_i \oplus \bigoplus_{h\in\Omega^i,h'=i} \mathbf{V}_{h''}) = r\}$$

form a partition of $\mathbf{E}_{\mathbf{V},\mathbf{W}^\bullet,\Omega^{i,N}}$. More precisely, $\mathbf{E}^r_{\mathbf{V},\mathbf{W}^\bullet,i}$ contains representations of the form $S_i^{\oplus r} \oplus M$.

Let $\mathcal{N}_{\mathbf{V},i}$ be the full subcategory of $\mathcal{D}^b_{G_\mathbf{V}}(\mathbf{E}_{\mathbf{V},\mathbf{W}^\bullet,\Omega^{i,N}})$ consisting of objects whose supports are contained in $\mathbf{E}^{\geqslant 1}_{\mathbf{V},\mathbf{W}^\bullet,i}$.

Fourier-Deligne transform and $\mathcal{N}_{\mathbf{V}}$

For two orientations Ω, Ω', Lusztig introduced the Fourier-Deligne transformation $\mathcal{F}_{\Omega,\Omega'}$ for quivers , which induces equivalences between $\mathcal{D}^b_{G_{\mathbf{V}}}(\mathbf{E}_{\mathbf{V},\Omega})$ and $\mathcal{D}^b_{G_{\mathbf{V}}}(\mathbf{E}_{\mathbf{V},\Omega'})$ and commutes with the induction functor

$$\mathcal{F}_{\Omega,\Omega'}(\mathbf{Ind}^{\mathbf{V}}_{\mathbf{V}',\mathbf{V}''}(L_1 \boxtimes L_2)) \cong \mathbf{Ind}^{\mathbf{V}}_{\mathbf{V}',\mathbf{V}''}(\mathcal{F}_{\Omega,\Omega'}(L_1) \boxtimes \mathcal{F}_{\Omega,\Omega'}(L_2)).$$

Apply Fourier-Deligne transform for framed quivers, for a given orientation Ω of Q, let Ω^N be the associated orientation of $Q^{(N)}$. Define $\mathcal{N}_{\mathbf{V}}$ to be the thick subcategory of $\mathcal{D}^b_{G_{\mathbf{V}}}(\mathbf{E}_{\mathbf{V},\mathbf{W}^\bullet,\Omega^N})$ generated by objects in $\mathcal{F}_{\Omega^{i,N},\Omega^N}(\mathcal{N}_{\mathbf{V},i}), i \in I$. We can consider the Verdier quotient $\mathcal{D}^b_{G_{\mathbf{V}}}(\mathbf{E}_{\mathbf{V},\mathbf{W}^\bullet,\Omega^N})/\mathcal{N}_{\mathbf{V}}$.

Localizations

For each $1 \leqslant k \leqslant N$, we set $\mathbf{E}_{\mathbf{V},\mathbf{W}^k,\Omega^N} = \mathbf{E}_{\mathbf{V},\Omega} \oplus \bigoplus_{i \in I} \mathbf{Hom}(\mathbf{V}_i, \mathbf{W}_{i^k})$, and let $\mathcal{Q}_{\mathbf{V},\mathbf{W}^k} = (\pi_{\mathbf{W}^k})^*(\mathcal{Q}_{\mathbf{V}})$ be the pull-back of $\mathcal{Q}_{\mathbf{V}}$ along the natural projection.

Definition

(1)Let $\mathcal{P}_{\mathbf{V},\mathbf{W}^\bullet}$ be the set of all simple direct summands (up to shifts) of $\mathbf{Ind}(A_1 \boxtimes A_2 \cdots \boxtimes A_N)$, where each A_k runs over objects in $\mathcal{Q}_{\mathbf{V}^k,\mathbf{W}^k}$ such that $\sum_{1 \leqslant k \leqslant N} |\mathbf{V}^k| = |\mathbf{V}|$.
(2) Let $\mathcal{Q}_{\mathbf{V},\mathbf{W}^\bullet}$ be the full subcategory consisting of finite direct sums of shifts of objects in $\mathcal{P}_{\mathbf{V},\mathbf{W}^\bullet}$.
(3)Let $\mathcal{Q}_{\mathbf{V},\mathbf{W}^\bullet}/\mathcal{N}_{\mathbf{V}}$ be the full subcategories of $\mathcal{D}^b_{G_{\mathbf{V}}}(\mathbf{E}_{\mathbf{V},\mathbf{W}^\bullet,\Omega^N})/\mathcal{N}_{\mathbf{V}}$ consisting of objects which are isomorphic to obejects of $\mathcal{Q}_{\mathbf{V},\mathbf{W}^\bullet}$. Denote $\coprod_{\mathbf{V}} \mathcal{Q}_{\mathbf{V},\mathbf{W}^\bullet}/\mathcal{N}_{\mathbf{V}}$ by $\mathcal{L}(\Lambda^\bullet)$.

Theorem

We can define endofunctors E_i, F_i and $K_i, i \in I$ of the localization $\mathcal{L}(\Lambda^\bullet)$. They satisfy the following relations

$$K_iK_j = K_jK_i,$$

$$E_iK_j = K_jE_i[-a_{j,i}],$$

$$F_iK_j = K_jF_i[a_{i,j}],$$

$$E_iF_j = F_jE_i \text{ for } i \neq j,$$

$$E_iF_i \oplus \bigoplus_{0 \leqslant m \leqslant N-1} Id[N-1-2m] \cong F_iE_i \oplus \bigoplus_{0 \leqslant m \leqslant -N-1} Id[-2m-N-1],$$

as endofunctors of $\mathcal{L}_{\mathbf{V}}(\Lambda^\bullet)$, where N is determined by dimension vector of $\mathbf{V}$ and $\Lambda^\bullet$.

Theorem

$$\bigoplus_{0\leqslant m\leqslant 1-a_{i,j}, m \ odd} E_i^{(m)} E_j E_i^{(1-a_{i,j}-m)}$$

$$\cong \bigoplus_{0\leqslant m\leqslant 1-a_{i,j}, m \ even} E_i^{(m)} E_j E_i^{(1-a_{i,j}-m)},$$

$$\bigoplus_{0\leqslant m\leqslant 1-a_{i,j}, m \ is \ odd} F_i^{(m)} F_j F_i^{(1-a_{i,j}-m)}$$

$$\cong \bigoplus_{0\leqslant m\leqslant 1-a_{i,j}, m \ is \ even} F_i^{(m)} F_j F_i^{(1-a_{i,j}-m)}.$$

The Grothendieck group $\mathcal{K}_0$

We define the Grothendieck group $\mathcal{K}_0(\Lambda^\bullet)$ of $\mathcal{L}(\Lambda^\bullet)$ to be the $\mathbb{Z}[v, v^{-1}]$-module spanned by objects in those $\mathcal{Q}_{\mathbf{V},\mathbf{W}^\bullet}/\mathcal{N}_{\mathbf{V}}$ with relations:

$$[X \oplus Y] = [X] + [Y],$$
$$[X[1]] = v[X].$$

The above theorem tells us that $\mathcal{K}_0(\Lambda^\bullet)$ is an integrable $\mathbf{U}$-module.

Module structure

Proposition

When $N = 1$, $\mathcal{K}_0(\Lambda)$ is canonically isomorphic to the integrable highest weight module $L(\Lambda)$, and its highest weight vector is the constant sheaf $[L_{\underline{d}^1}]$ on the moduli space of 0-dimension.

Proposition

The functor $P\mathbf{D} \bigoplus_{\mathbf{V}^1,\mathbf{V}^2} \mathbf{Res}^{\mathbf{V}\oplus\mathbf{W}^\bullet}_{\mathbf{V}^1\oplus\mathbf{W}^1,\mathbf{V}^2\oplus\mathbf{W}^2}\mathbf{D}$ induces isomorphisms of ${}_{\mathcal{A}}\mathbf{U}$-modules

$$\Delta_N : \mathcal{K}_0(\Lambda^\bullet) \to \mathcal{K}_0(\Lambda_N) \otimes \mathcal{K}_0(\Lambda^{\bullet-1}).$$

Main Theorem

By induction on N, we can prove

Theorem

For sequences of graded space $\mathbf{W}^\bullet$ and dominant weight $\Lambda^\bullet$ such that $\langle \Lambda_k, \alpha_i^\vee \rangle = \dim \mathbf{W}_{i^k}$, the Grothendieck group $\mathcal{K}_0(\Lambda^\bullet)$ together with the functors $E_i^{(n)}, F_i^{(n)}, K_i^\pm, i \in I, n \in \mathbb{N}$ becomes a ${}_\mathcal{A}\mathbf{U}$-module, which is isomorphic to tensor products of highest weight modules ${}_\mathcal{A}L(\Lambda_N) \otimes {}_\mathcal{A}L(\Lambda_{N-1}) \otimes \cdots \otimes {}_\mathcal{A}L(\Lambda_1)$.

Question: Are these simple objects in localization provide the canonical basis b of irreducible modules or $b \diamond b'$ of tensor products?

The canonical bases of certain tensor products

Recall that the quasi $\mathcal{R}$-matrix $\Theta = \sum_{\nu \in \mathbb{N}I} ((-1)^{tr\nu} v_\nu \sum_{b \in \mathcal{B}_\nu} b^+ \otimes b^{*-})$ is a well defined operator on tensor products of integrable modules $M \otimes M'$. Let $\Psi(m \otimes m') = \Theta(\bar{m} \otimes \bar{m}')$, then $\Psi u = \bar{u}\Psi$ for any $u \in \mathbf{U}$. Assume $\mathcal{B}_1$ and $\mathcal{B}_2$ are canonical bases of $L(\Lambda_1)$ and $L(\Lambda_2)$, then for any $b_i \in \mathcal{B}_i$, there exists a unique element $b_1 \diamond b_2$ in $L(\Lambda_1) \otimes L(\Lambda_2)$ such that

$$\Psi(b_1 \diamond b_2) = b_1 \diamond b_2,$$

$$b_1 \diamond b_2 = b_1 \otimes b_2 \mod v^{-1}\mathbf{L},$$

where $\mathbf{L}$ is a $\mathbb{Z}[v^{-1}]$-submodule defined by a bilinear form on $L(\Lambda_1) \otimes L(\Lambda_2)$. Those $b_1 \diamond b_2$ form the canonical basis of $L(\Lambda_1) \otimes L(\Lambda_2)$.

Geometric $\mathcal{R}$-matrix

Recall that the restriction functor gives an isomorphism $\Delta_2 : \mathcal{K}_0(\Lambda_1, \Lambda_2) \to \mathcal{K}_0(\Lambda_2) \otimes \mathcal{K}_0(\Lambda_1)$, we can define

$$\Psi' = \Delta_2 \mathbf{D} \Delta_2^{-1},$$

$$\Theta' = \Delta_2 \mathbf{D} \Delta_2^{-1} (\mathbf{D} \otimes \mathbf{D}).$$

Proposition

$$\Theta' = \Theta, \Psi' = \Psi : \mathcal{K}_0(\Lambda_2) \otimes \mathcal{K}_0(\Lambda_1) \to \mathcal{K}_0(\Lambda_2) \otimes \mathcal{K}_0(\Lambda_1).$$

Geometric bilinear form

Definition

Define a bilinear form $(-,-)^{\Lambda^\bullet}$ on $\mathcal{K}_0(\Lambda^\bullet)$ by:

$$([A],[B])^{\Lambda^\bullet} = \sum_{n\geq 0} \mathrm{dim}\mathrm{Ext}^n_{\mathcal{D}^b_{G_{\mathbf{V}}}(\mathbf{E}_{\mathbf{V},\mathbf{W}^\bullet,\Omega^i,N})/\mathcal{N}_{\mathbf{V}}}(DA,B)v^{-n},$$

for any objects A,B of $\mathcal{L}_{\mathbf{V}}(\Lambda^\bullet)$. Otherwise, if A is a object of $\mathcal{L}_{\mathbf{V}}(\Lambda^\bullet)$ and B is a object of $\mathcal{L}_{\mathbf{V}'}(\Lambda^\bullet)$ such that $|\mathbf{V}|\neq|\mathbf{V}'|$, define

$$([A],[B])^{\Lambda^\bullet} = 0.$$

Then by adjointness, we can show

$$([F_iA],[B])^{\Lambda^\bullet} = ([A],v[K_i^{-1}E_iB])^{\Lambda^\bullet}.$$

Almost orthogonal

Proposition

(1)If $[L] \neq [L']$, *then* $([L],[L'])^{\Lambda^\bullet} \in v^{-1}\mathbb{Z}[[v^{-1}]] \cap \mathbb{Q}(v)$ *;*
(2)If L *is a nonzero simple perverse sheaf in* $\mathcal{L}_{\mathbf{V}}(\Lambda^\bullet)$, *then* $([L],[L])^{\Lambda^\bullet} \in 1 + v^{-1}\mathbb{Z}[[v^{-1}]] \cap \mathbb{Q}(v)$ *.*

In a conclusion, our basis is invariant under the involution induced by $\mathbf{D}$, and almost-orthogonal with respect to the geometric bilinear form.

Compatible with Δ_2

Notice that $(-,-)^{\Lambda^\bullet}$ defines a contravariant bilinear form on $\mathcal{K}_0(\Lambda_1,\Lambda_2)$, and $(m_2\otimes m_1, n_2\otimes n_1)^{\otimes} = (m_2,n_2)^{\Lambda_2}(m_1,n_1)^{\Lambda_1}$ defines a bilinear form $(-,-)^{\otimes}$ on $\mathcal{K}_0(\Lambda_2)\otimes\mathcal{K}_0(\Lambda_1)$,

Proposition

For any x,y in $\mathcal{K}_0(\Lambda_1,\Lambda_2)$,

$$(x,y)^{\Lambda^\bullet} = (\Delta_2(x),\Delta_2(y))^{\otimes}.$$

Theorem

$\{\Delta_2([L]) | L$ is a simple perverse sheaf in localization$\}$ form a Ψ-invariant and almost orthogonal basis of $\mathcal{K}_0(\Lambda_2) \otimes \mathcal{K}_0(\Lambda_1)$. In particular, it coincides with the canonical basis of tensor products.

Corollary

If the quiver is of finite type ADE, then the structure coefficients of irreducible integrable lowest (and highest) modules under the action of elements of canonical basis of $\dot{\mathbf{U}}$ with respect to the canonical basis of irreducible modules are in $\mathbb{N}[v, v^{-1}]$,

$$(b_1 \diamond_\zeta b_2)(b^+ v_{-\Lambda}) \in \sum_{b'} \mathbb{N}[v, v^{-1}](b'^+ v_{-\Lambda}).$$

Yang-Baxter equation

Repalce $\mathbf{Res}^{\mathbf{V}\oplus\mathbf{W}^\bullet}_{\mathbf{V}^1\oplus\mathbf{W}^1,\mathbf{V}^2\oplus\mathbf{W}^2}$ by $\mathbf{Res}^{\mathbf{V}\oplus\mathbf{W}^\bullet}_{\mathbf{V}^1\oplus\mathbf{W}^2,\mathbf{V}^2\oplus\mathbf{W}^1}$, one can similar defines a $\mathbf{U}$-isomorphism

$$\Delta'_N : \mathcal{K}_0(\Lambda^\bullet) \to \mathcal{K}_0(\Lambda^{\bullet-1}, 0) \otimes \mathcal{K}_0(0, \Lambda_N),$$

then $\Delta'_2 \circ \Delta_2^{-1} : \mathcal{K}_0(\Lambda_2) \otimes \mathcal{K}_0(\Lambda_1) \to \mathcal{K}_0(\Lambda_1) \otimes \mathcal{K}_0(\Lambda_2)$.

Proposition

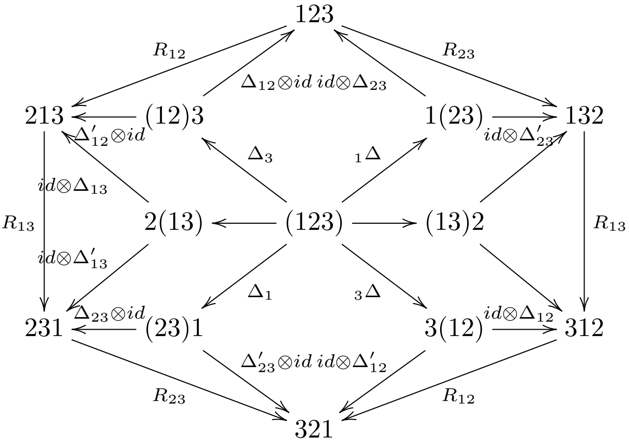
We have $\Delta'_2 \circ \Delta_2^{-1} = P\Theta^{\tilde{f}}$ *for* $f(\alpha, \beta) = v^{-(\alpha,\beta)}$.

Now we assume $N = 3$ and let R_{ij} be the following isomorphism of $\mathbf{U}$-modules

$$\Delta_2' \circ \Delta_2^{-1} : L(\Lambda_i) \otimes L(\Lambda_j) \to L(\Lambda_j) \otimes L(\Lambda_i).$$

Regarding $\mathcal{K}_0(\Lambda_i)$ as the integrable highest weight module ${}_{\mathcal{A}}L(\Lambda_i)$, we have the following Yang-Baxter equation of isomorphisms beween ${}_{\mathcal{A}}\mathbf{U}$-modules

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23} :$$
$$L(\Lambda_1) \otimes L(\Lambda_2) \otimes L(\Lambda_3) \xrightarrow{\cong} L(\Lambda_3) \otimes L(\Lambda_2) \otimes L(\Lambda_1).$$



Coassociativity of Hall algebra implies Yang-Baxter equation!

Thanks!