Mutation of Brauer configuration algebras

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1. T. Aihara and O. Iyama. Silting mutation in triangulated categories. *J. Lond. Math. Soc. (2)*, 85(3):633–668, 2012.
2. T. Aihara. Mutating Brauer trees. *Math. J. Okayama Univ.*, 56:1–16, 2014.
3. T. Aihara. Derived equivalences between symmetric special biserial algebras. *J. Pure Appl. Algebra*, 219(5):1800–1825, 2015.

Mutation:
Brauer tree algebras(BTA): $\surd$ see 2. finite-representation type
Brauer graph algebras(BGA): $\surd$ see 3. tame-representation type
Aim of this talk: Mutation of Brauer configuration algebras(BCA).
BCA are mostly wild-representation type

Let A be a finite dimensional algebra over algebraically closed field K.

Definition

Let $T = (T^i, d^i)$ be a bounded complex of $\mathrm{K}^{\mathrm{b}}(\operatorname{proj} A)$.

1. We say that T is *presilting* (resp., *pretilting*) if $\operatorname{Hom}(T, T[i]) = 0$ for all integer $i > 0$ (resp., $i \neq 0$).
2. We say that T is *silting* (resp., *tilting*) if it is presilting (resp., pretilting) and thick $T = \mathrm{K}^{\mathrm{b}}(\operatorname{proj} A)$, where thick T is the smallest triangulated full subcategory of $\mathrm{K}^{\mathrm{b}}(\operatorname{proj} A)$ which is closed under taking direct summands.
3. We say that T is *two-term* if $T^i = 0$ for all $i \neq 0, -1$.

Definition-Proposition (Aihara and Iyama)

Let $T = X \oplus Q$ *be a basic silting complex in* $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\, A)$ *with an indecomposable direct summand* X*. We take a triangle*

$$X \xrightarrow{f} Q' \longrightarrow Y \longrightarrow X[1],$$

where f *is a left minimal* $(\mathrm{add}\, Q)$*-approximation of* X*. Then,* Y *is indecomposable. In this case,* $\mu_X^-(T) := Y \oplus Q$ *is again a basic silting complex and called left mutation of* T *with respect to* X*. The right mutation* $\mu_X^+(T)$ *is defined dually.*

Example

$\forall$ indec proj A-module P, the left mutation $\mu_P^-(A)$ is silting. If it is tilting, then its endomorphism algebra $\mathrm{End}(\mu_P^-(A))$ is known as *tilting mutation* of the algebra A. Notice that it is derived equivalent to A.

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Definition (Green and Schroll)

Let $\Gamma := (H, \sigma, \psi, s, \mathfrak{m})$ be a tuple, where H is a non-empty finite set, σ is a permutation on H, and ψ is an equivalent relation on H each of whose equivalence classes has at least two elements.

(1) Each element of H is called an *angle* of Γ.

(2) Each equivalence class in H/ψ is called a *polygon* of Γ.

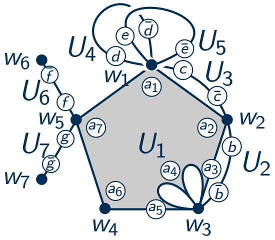
(3) Let $s\colon H \to H/\langle\sigma\rangle$ be a canonical surjection. Each element of $H/\langle\sigma\rangle$ is called a *vertex* of Γ.

(4) For each vertex u of Γ, the σ-orbit $(h, \sigma(h), \ldots, \sigma^{\mathrm{val}(u)-1}(h))$ incident to u is called the *cyclic ordering* around u.

(5) $\mathfrak{m}\colon H/\langle\sigma\rangle \to \mathbb{Z}_{>0}$ is a function which we call the *multiplicity function*.

Example

Angels $H := \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, b, \bar{b}, c, \bar{c}, d, \bar{d}, e, \bar{e}, f, \bar{f}, g, \bar{g}\}$.
Polygons $H/\psi := \{U_1 \ldots, U_7\}$. Such as, $U_1 := \{a_1, \ldots, a_7\}$ is a 7-gon.
Vertices $H/\sigma := \{w_1 \ldots, w_7\}$. cyclic orderings are counterclockwise.
Such as, $w_1 = (a_1, c, \bar{e}, \bar{d}, e, d)$.
multiplicity function $\mathfrak{m}(w_4) = 2$ and $\mathfrak{m}(w_i) = 1$ for all $i \neq 4$.



Quiver

Definition

Let $\Gamma := (H, \sigma, \psi, s, \mathfrak{m})$ be a Brauer configuration. Let Q_Γ be a finite quiver defined as follows:

- The set of vertices is the set H/ψ of polygons of Γ.
- The set of arrows is in bijection with the set H of angles of Γ, where we draw an arrow $[h] \to [\sigma(h)]$ for every $h \in H$. We write this arrow by the same symbol h or $\sigma^0(h)$ if there is no confusion.

For given $h, f \in H$ such that $f = \sigma^m(h)$ for some $1 \leq m \leq \mathrm{val}(s(h))$, let $C_{h,f}$ be a path

$$[h] \xrightarrow{\sigma^0(h)} [\sigma(h)] \xrightarrow{\sigma(h)} [\sigma^2(h)] \longrightarrow \cdots \longrightarrow [\sigma^{m-1}(h)] \xrightarrow{\sigma^{m-1}(h)} [f]$$

of length m in the quiver Q_Γ. We have a cycle $C_h := C_{h,h}$ of length $\mathrm{val}(s(h))$.

Relations

Let I_Γ be an ideal in the path algebra KQ_Γ generated by all the relations in (BC1) and (BC2):

(BC1) For any polygon V and any $h, f \in V$,

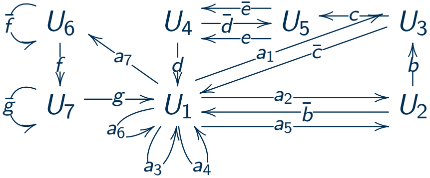
$$C_h^{\mathfrak{m}(s(h))} - C_f^{\mathfrak{m}(s(f))}.$$

(BC2) All paths in Q_Γ of length two which are not sub-paths of $C_h^{\mathfrak{m}(s(h))}$ for any $h \in H$.

We define $A_\Gamma := KQ_\Gamma / I_\Gamma$ and call it *Brauer configuration algebra* of Γ.

Example

Let Γ be a BC given in above. We obtain a BCA $A_\Gamma = kQ_\Gamma / I_\Gamma$, where Q_Γ is the quiver given by



and I_Γ is the ideal generated by the following relations:

- $a_1c\bar{e}\bar{d}ed = a_2b\bar{c} = a_3a_4a_5\bar{b} = a_4a_5\bar{b}a_3 = a_5\bar{b}a_3a_4 = a_6^2 = a_7fg$, $b\bar{c}a_2 = \bar{b}a_3a_4a_5$, $c\bar{e}\bar{d}eda_1 = \bar{c}a_2b$, $da_1c\bar{e}\bar{d}e = \bar{d}eda_1c\bar{e}$, $eda_1c\bar{e}\bar{d} = \bar{e}\bar{d}eda_1c$, $fga_7 = \bar{f}$ and $ga_7f = \bar{g}$.
- All paths of length 2 which are not sub-paths of the above monomials.

BGA
Tilting mutation of Brauer graph algebras is compatible with flip of Brauer graphs (Aihara).
The class of Brauer graph algebras is closed under derived equivalence (Antipov and Zvonareva).

BCA
Tilting mutation $\mathrm{End}(\mu^{-}_{P_V}(A_\Gamma))$ is not a Brauer configuration algebra in general. Hence, the class of Brauer configuration algebras is not closed under derived equivalence (Aoki and Zhang).

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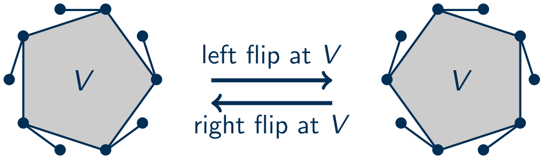
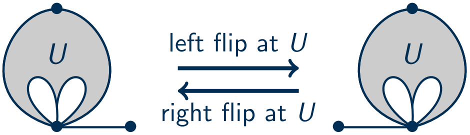
Definition (Aoki and Zhang)

We say that a polygon V of Γ satisfies the condition (E) if every predecessor of V in the cyclic ordering around each vertex is either an edge or V itself. In this case, we define a new Brauer configuration $\mu_V^-(\Gamma)$, which we call a (left) *flip* of Γ at V. Dually, one can define a right flip. For more details, see the following

- T. Aoki and Y. Zhang. Mutation of Brauer configuration algebras, arXiv:2403.14134.

Flip of BC


V
left flip at V
right flip at V
V


U
left flip at U
right flip at U
U

Theorem (Aoki and Zhang)

Let Γ be a Brauer configuration and A_Γ the Brauer configuration algebra of Γ over an algebraically closed field K. If a polygon V of Γ satisfies the condition (E), then there is an isomorphism of K-algebras

$$\mathrm{End}(\mu^-_{P_V}(A_\Gamma)) \cong A_{\mu^-_V(\Gamma)}.$$

In particular, it is again a Brauer configuration algebra.

Under the condition (E), flip of Brauer configurations is compatible with tilting mutation of the corresponding Brauer configuration algebras.

$$
\begin{array}{ccc}
BC & \longrightarrow & BCA \\
 & & \\
\Gamma & \longrightarrow & A_\Gamma \\
\Big\downarrow \text{flip} & & \Big\downarrow \text{tilting mutation} \\
\mu_V^-(\Gamma) & \longrightarrow & A_{\mu_V^-(\Gamma)} \cong \mathrm{End}(\mu_{P_V}^-(A_\Gamma))
\end{array}
$$

End

Thank You