Model structures and Q-shaped derived category

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- Model structures were introduced by Quillen in 1967, and are mostly motivated by their utility in solving a localization problem for categories.
- Abelian model structures were introduced by Hovey in 2002. It's related to relative homological algebra and triangulated categories.

1. Model structures
2. Abelian model structures on $\mathsf{Ch}(R)$
3. Abelian model structures on ${}_{\mathcal{Q},R}\mathsf{Mod}$
4. Flat model structures on ${}_{\mathcal{Q},R}\mathsf{Mod}$

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Setup

R: associative ring — E: bicomplete category — $\mathcal{A}$: bicomplete abelian category

Model structures

- Given morphisms $f : A \to B$, $g : C \to D$ morphisms in E, we say that the pair (f, g) is orthogonal, denoted by $f \perp g$, if for any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{l} & C \\ {\scriptstyle f}\downarrow & \overset{t}{\nearrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[r]{} & D \end{array}$$

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  $\exists\ t : B \to C$ such that both the triangles commute.
  Thus we say f has the left lifting property for g and g has the right lifting property for f.

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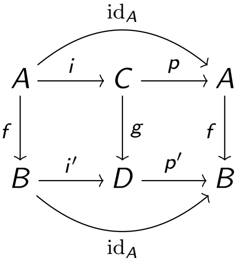
- A morphism $f : A \to B$ in E is said to be a retract of a morphism $g : C \to D$ in E if f is a retract of g as objects of the category of morphisms in E. I.e., $\exists\ i, i', p, p'$ with $pi = \mathrm{id}_A$ and $p'i' = \mathrm{id}_B$ s.t. the following diagram commutes:



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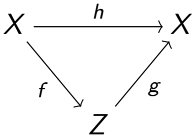
- $\mathcal{C}$ and $\mathcal{F}$ are closed under retracts.
- The pair (f, g) is orthogonal for all $f \in \mathcal{C}$ and $g \in \mathcal{F}$.
- For any morphism $h : X \to Y$ in E, $\exists$ a factorization



with $f \in \mathcal{C}$ and $g \in \mathcal{F}$.

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$$\mathsf{C}^{\perp} = \{N \in \mathsf{A} \mid \operatorname{Ext}^1_{\mathsf{A}}(C, N) = 0 \text{ for all } C \in \mathsf{C}\},$$

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- A cotorsion pair (C,F) in $\mathcal{A}$ is called complete if for each object M in A, there are two following exact sequence:

$$0\to M\to F\to C\to 0 \text{ and } 0\to F'\to C'\to M\to 0$$

with $C,C'\in\mathsf{C}$ and $F,F'\in\mathsf{F}$.

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$$\mathrm{Mon}(\mathsf{C}) = \left\{\alpha \;\middle|\; \alpha \text{ is a monomorphism with } \operatorname{Coker}\alpha \in \mathsf{C} \right\},$$

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Definition (Quillen, 1967)

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- A model category is a bicomplete category with a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

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- A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{A}$ is called abelian if $\mathcal{C}$ is the class of all monomorphisms with cofibrant cokernels and $\mathcal{F}$ is the class of all epimorphisms with fibrant kernels.

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  That is, an abelian model structure is a model structure compatible with abelian category.

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Such a triple is called a Hovey triple.

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- Given a Hovey triple $(\mathsf{C}, \mathsf{W}, \mathsf{F})$, $\exists$ an abelian model structure $(\mathrm{Mon}(\mathsf{C}), \mathcal{W}, \mathrm{Epi}(\mathsf{F}))$, where $\mathcal{W} = \{w \mid w = fc \text{ with } c \in \mathrm{Mon}(\mathsf{C} \cap \mathsf{W}), f \in \mathrm{Epi}(\mathsf{W} \cap \mathsf{F})\}$.

So the problem of constructing model structures is transformed into that of how to construct Hovey triples.

Abelian model structures on Ch(R)

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ABELIAN MODEL STRUCTURES ON $\mathsf{Ch}(R)$

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THEOREM (HOVEY, 2002)

For any ring R, $\exists$ two hereditary abelian model structures on $\mathsf{Ch}(R)$ as follows: The proj. model structure $(\mathsf{dgP}, \mathscr{E}, \mathsf{Ch})$, and the inj. model structure $(\mathsf{Ch}, \mathscr{E}, \mathsf{dgI})$; their homotopy categories are derived category $\mathsf{D}(R)$.

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Theorem (Gillespie, 2004)

For any ring R, $\exists$ a hereditary abelian model structures on $\mathsf{Ch}(R)$: *The flat model structure* $(\mathsf{dgF}, \mathscr{E}, \mathsf{dgC} = \mathsf{dwC})$.

Abelian model structures on ${}_{\mathcal{Q},R}\mathsf{Mod}$

Setup

$\Bbbk$ is a commutative ring, R is a $\Bbbk$-algebra and $\mathcal{Q}$ is a small $\Bbbk$-linear category. Set ${}_{\mathcal{Q},R}\mathsf{Mod} = \{\Bbbk\text{-linear functors } \mathcal{Q} \to {}_R\mathsf{Mod}\}$ *and* ${}_{\mathcal{Q}}\mathsf{Mod} = \{\Bbbk\text{-linear functors } \mathcal{Q} \to {}_{\Bbbk}\mathsf{Mod}\}$

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Let's look at some examples of functor categories.

EXAMPLE

Consider the linear quiver Γ with the relations that consecutive arrows compose to zero where

$$\Gamma = \cdots \to \underset{-2}{\bullet} \xrightarrow{\partial} \underset{-1}{\bullet} \xrightarrow{\partial} \underset{0}{\bullet} \xrightarrow{\partial} \underset{1}{\bullet} \xrightarrow{\partial} \underset{2}{\bullet} \to \cdots$$

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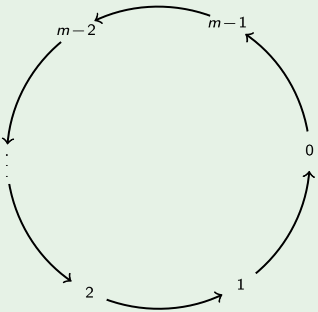
- Let $\mathcal{Q}$ be the path category of Γ. Then $\mathrm{Ch}(R) \simeq {}_{\mathcal{Q},R}\mathrm{Mod}$

Example

Consider the cyclic quiver with m vertices and the relations that consecutive arrows compose to 0.



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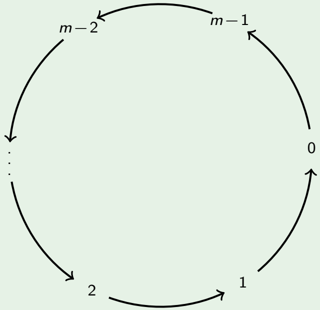

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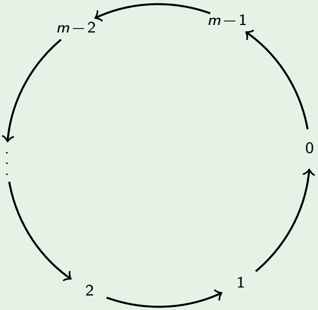

Let $\mathcal{Q}$ be the path category of the quiver. Then ${}_{\mathcal{Q},R}\mathrm{Mod}$ can be identified with the category of m-periodic complexes.

Abelian model structures on ${}_{\mathcal{Q},R}\mathsf{Mod}$

Question

Let $\mathcal{Q}$ be a small pre-additive category. Can we find projective, injective and flat model structures on ${}_{\mathcal{Q},R}\mathsf{Mod}$?

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If the $\mathcal{Q}$ is nice, then the answer is Yes.

We let

$$\mathscr{E} = \{\textit{exact complexes}\}$$

- A projective model structure on $\mathrm{Ch}(R)$ corresponds to $\mathscr{E}$ s.t. $\mathscr{E}$ is thick, $({}^{\perp}\mathscr{E}, \mathscr{E})$ is a complete cotorsion pair, and ${}^{\perp}\mathscr{E} \cap \mathscr{E} = \mathrm{Prj}$.

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- An injective model structure on $\mathrm{Ch}(R)$ corresponds to $\mathscr{E}$ s.t. $\mathscr{E}$ is thick, $(\mathscr{E}, \mathscr{E}^{\perp})$ is a complete cotorsion pair, and $\mathscr{E} \cap \mathscr{E}^{\perp} = \mathsf{Inj}$.

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- Figure out how to define the class $\mathscr{E}$ of exact objects in functor category ${}_{\mathcal{Q},R}\mathrm{Mod}$ and prove that $\mathscr{E}$ satisfies all the required conditions above.

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Fact

$\mathrm{Ch}(\mathbb{Z})$ is a locally Gorenstein category.

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FACT

$\mathrm{Ch}(\mathbb{Z})$ is a locally Gorenstein category.

Recall that an abelian category $\mathcal{A}$ is locally Gorenstein if

- For any $M \in \mathcal{A}$ one has $\mathrm{pd}_{\mathcal{A}} M < \infty \Leftrightarrow \mathrm{id}_{\mathcal{A}} M < \infty$.
- $\mathrm{FPD}(\mathcal{A})$ and $\mathrm{FID}(\mathcal{A})$ are both finite.
- $\mathcal{A}$ has a generator of finite proj. dim.

Question

For which $\mathcal{Q}$ is ${}_{\mathcal{Q}}\mathsf{Mod}$, the additive functors from $\mathcal{Q}$ to ${}_{\Bbbk}\mathsf{Mod}$, locally Gorenstein?

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Theorem (Dell'Ambrogio, Stevenson, Šťovíček, 2017)

If $\Bbbk$ is a Gorenstein ring and $\mathcal{Q}$ is Gorenstein, then ${}_{\mathcal{Q}}\mathrm{Mod}$ is locally Gorenstein.

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- existence of a Serre functor: $\exists$ a $\mathbb{k}$-linear autoequivalence $\mathbb{S} : \mathcal{Q} \to \mathcal{Q}$ and a natural isomorphism $\mathcal{Q}(p,q) \cong \mathrm{Hom}_{\mathbb{k}}(\mathcal{Q}(q,\mathbb{S}(p)),\mathbb{k})$.

Abelian model structures on ${}_{\mathcal{Q},R}\mathrm{Mod}$

Recall that a small pre-additive $\Bbbk$-category $\mathcal{Q}$ is Gorenstein provided that

- Hom-finite: Each hom set $\mathcal{Q}(p,q)$ is f.g. proj. $\Bbbk$-module.
- locally bounded: For each $q \in \mathcal{Q}$, the two sets

  $\mathrm{N}_-(q) = \{p \mid \mathcal{Q}(p,q) \neq 0\}$ and $\mathrm{N}_+(q) = \{r \mid \mathcal{Q}(q,r) \neq 0\}$

  are finite.
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- strong retraction property : For each $q \in \mathcal{Q}$, $\exists$ $\Bbbk$-module decomposition $\mathcal{Q}(q,q) = (\Bbbk \cdot \mathrm{id}_q) \oplus \mathfrak{r}_q$ and $\mathfrak{r}_q \circ \mathfrak{r}_q \subseteq \mathfrak{r}_q$ for all q, and $\mathcal{Q}(q,p) \circ \mathcal{Q}(p,q) \subseteq \mathfrak{r}_p$ for all $p \neq q$.

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The category $\mathcal{Q}$ is often defined by path category of a quiver with relations.

Example

Consider the quiver Γ with the relations $\partial^2 = 0$ where

$$\Gamma = \cdots \to \underset{2}{\bullet} \xrightarrow{\partial} \underset{1}{\bullet} \xrightarrow{\partial} \underset{0}{\bullet} \xrightarrow{\partial} \underset{-1}{\bullet} \xrightarrow{\partial} \underset{-2}{\bullet} \to \cdots$$

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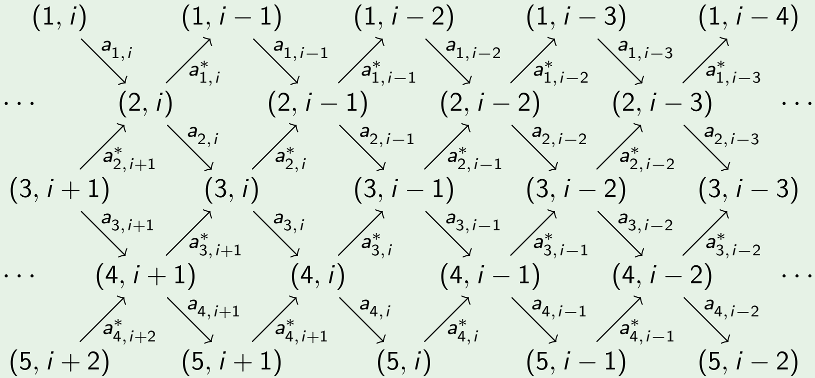
- $\mathbb{S}$(q)=q-1.

Example

Consider the repetitive quiver $\mathbb{Z}A_5$ of A_5



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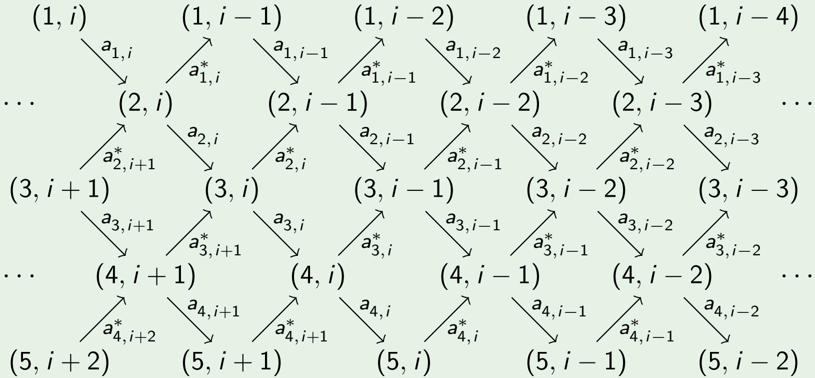

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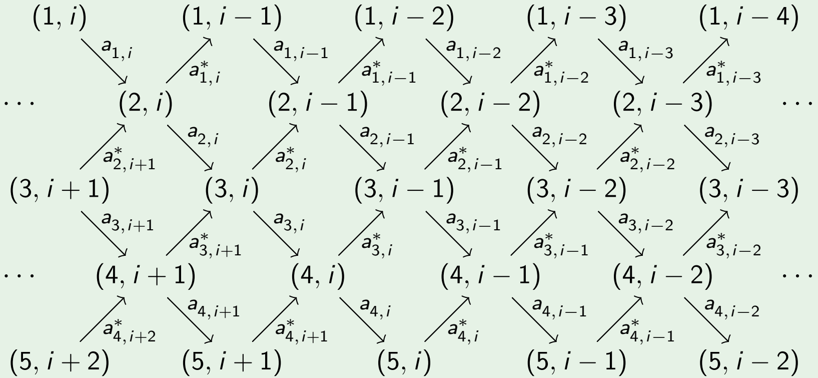

Let Γ be the repetitive quiver $\mathbb{Z}A_5$ modulo mesh relations and Q the path category of Γ. Then $\mathcal{Q}$ is a Gorenstein category.

- Serre functor $\mathbb{S}$ is defined by $\mathbb{S}(p, i) = (6 - p, i + 1 - p)$

Abelian model structures on ${}_{\mathcal{Q},R}\mathsf{Mod}$

Setup

$\mathcal{Q}$: *Gorenstein small pre-additive* $\Bbbk$*-category*

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AIM: Figure out how to define the class $\mathscr{E}$ of exact objects in ${}_{\mathcal{Q},R}\mathsf{Mod}$ such that $\mathscr{E}$ satisfies all of the required conditions.

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AIM: Figure out how to define the class $\mathscr{E}$ of exact objects in ${}_{\mathcal{Q},R}\mathsf{Mod}$ such that $\mathscr{E}$ satisfies all of the required conditions.

Definition (Holm and Jørgensen, 2022)

Let $\mathbb{k}$ *be a Gorenstein ring. Define*

$$\mathscr{E} = \{X \in {}_{\mathcal{Q},R}\mathsf{Mod} \mid \mathrm{pd}(X^{\natural}) \textit{ or } \mathrm{id}(X^{\natural}) \textit{is finite in } {}_{\mathcal{Q}}\mathsf{Mod}\}$$

Here $(-)^{\natural} : {}_{\mathcal{Q},R}\mathsf{Mod} \to {}_{\mathcal{Q}}\mathsf{Mod}$ *is the forgetful functor.*

Abelian model structures on ${}_{\mathcal{Q},R}\mathsf{Mod}$

If $\mathcal{Q}$ is Gorenstein, then $\mathscr{E}$ satisfies all the required conditions, Holm and Jørgensen constructed the two hereditary abelian model structres on functor category ${}_{\mathcal{Q},R}\mathsf{Mod}$.

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Let $\mathbb{k}$ be Gorenstein. Then $\exists$ two hereditary abelian model structures on ${}_{\mathcal{Q},R}\mathrm{Mod}$ as follows:

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Remark

If $\mathcal{Q}$ is the path category of linear quiver with the relation that the consecutive arrows compose to 0, then $\mathcal{Q}$-shaped derived category is the usual derived category.

Flat model structures on ${}_{\mathcal{Q},R}\mathsf{Mod}$

Theorem

For any ring R, $\exists$ a hereditary abelian model structures on $\mathsf{Ch}(R)$: The flat model structure $(\mathsf{dgF}, \mathscr{E}, \mathsf{dgC} = \mathsf{dwC})$.

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Can we construct flat model structures on functor categories?

FLAT MODEL STRUCTURES ON ${}_{\mathcal{Q},R}\mathsf{Mod}$

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Can we construct flat model structures on functor categories?

We can solve it by *PGF*-modules.

DEFINITION (ŠAROCH AND ŠT'OVÍČEK, 2020)

An object X on a category ${}_{\mathcal{Q}}\mathsf{Mod}$ is a PGF-module if $\exists$ an exact sequence

$$\cdots \to P^{-1} \xrightarrow{d} P^0 \to P^1 \to P^2 \to \cdots$$

with each P^i projective such that $X = \mathrm{Coker} d$ and the sequence remains exact after tensor by any injecive object in $\mathsf{Mod}_{\mathcal{Q}}$.

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Flat model structures on ${}_{\mathcal{Q},R}\mathrm{Mod}$

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- $(\mathsf{PGF}, \mathsf{PGF}^{\perp})$ *is a proj. cotorsion pair on* ${}_{\mathcal{Q}}\mathrm{Mod}$.

In what follows, we define the class

$$\mathsf{E} = \{\mathsf{X} \in {}_{\mathcal{Q},\mathsf{R}}\mathrm{Mod} \mid \mathsf{X}^{\natural} \in \mathsf{PGF}^{\perp}\}.$$

Flat model structures on ${}_{\mathcal{Q},R}\mathsf{Mod}$

Theorem (Di, Li, Liang and Ma, 2023)

$\exists$ *a hereditary abelian model structures on* ${}_{\mathcal{Q},R}\mathsf{Mod}$*: The flat model structure* $({}^{\perp}(\mathsf{Cot}({}_{\mathcal{Q},R}\mathsf{Mod}) \cap \mathsf{E}), \mathsf{E}, \mathsf{Cot}({}_{\mathcal{Q},\mathsf{R}}\mathsf{Mod}))$*;*

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Remark

If $\Bbbk$ is Gorenstein, then ${}_{\mathcal{Q}}\mathsf{Mod}$ is a locally Gorenstein category. In this case, E is the class of exact objects and the above flat model structure's homotopy category is also the $\mathcal{Q}$-shaped derived category.

Model structures and Q-shaped derived category

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Lanzhou Jiaotong University

Zhenxing Di, Liping Li, Li Liang and Yajun Ma, Flat model structures and Gorenstein objects in functor categories, **Proc. Roy. Soc. Edinburgh Sect. A**, http://doi.org/10.1017/prm.2024.60.