

The lax functoriality of Hochschild cochain complex

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Motivations

The functoriality of Hochschild chain complex/homology

- ▶ The Hochschild chain complex is a **functor** $C_* : k\text{-alg} \longrightarrow \mathcal{C}k$

$$\begin{array}{ccccccc}
 A & & C_*(A) : \dots & \xrightarrow{d_n} & A^{\otimes n} & \xrightarrow{d_{n-1}} & \dots & \xrightarrow{d_2} & A^{\otimes 2} & \xrightarrow{d_1} & A \\
 \downarrow f & \longmapsto & \downarrow C_*(f) & & \downarrow f^{\otimes n} & & & & \downarrow f^{\otimes 2} & & \downarrow f \\
 B & & C_*(B) : \dots & \xrightarrow{d_n} & B^{\otimes n} & \xrightarrow{d_{n-1}} & \dots & \xrightarrow{d_2} & B^{\otimes 2} & \xrightarrow{d_1} & B
 \end{array}$$

- ▶ The Hochschild homology is a **functor**

$$HH_* = H_* \circ C_* : k\text{-alg} \longrightarrow \mathbf{Gr}k.$$

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$$HH_* = H_* \circ C_* : k\text{-alg} \longrightarrow \mathbf{Gr}k.$$

The Hochschild cochain complex/cohomology are not functors

The Hochschild cochain complex of k -algebras is **not** a functor.

The Hochschild cohomology of k -algebras is **not** a functor.

The center of k -algebras is **not** a functor.

Example

Let $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, $S = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$. Then $Z(A) \cong k$, $Z(S) = S \cong k^2$.

$$\begin{array}{ccc}
 S & \xrightarrow{\text{id}} & S \\
 \curvearrowright & & \curvearrowright \\
 A & \longrightarrow & S
 \end{array}
 \implies
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The lax functoriality of center

Theorem (Grady-Oren 2021)

*The center of k -algebras is a **lax functor**.*

Q1: Can we extend the lax functoriality of center to Hochschild cochain complex?

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An inspiring work of Keller

Theorem (Keller2003)

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be *dg categories* cofibrant over k ,
 and ${}_{\mathcal{A}}X_{\mathcal{B}}, {}_{\mathcal{B}}Y_{\mathcal{C}}$ are *cofibrant dg bimodules*,
 such that $- \otimes_{\mathcal{A}}^{\mathbb{L}} X : \text{per } \mathcal{A} \rightarrow \mathcal{DB}$ is *fully faithful*.

Then there is a morphism $\mathcal{C}(X) : C(\mathcal{B}) \rightarrow C(\mathcal{A})$ in $\text{Ho } B_{\infty}$.

- ▶ Under more conditions, $\mathcal{C}(X \otimes_{\mathcal{B}}^{\mathbb{L}} Y) = \mathcal{C}(X) \circ \mathcal{C}(Y)$. ♣ Preserves compositions.
- ▶ $\mathcal{C}(I_{\mathcal{A}}) = \text{id}_{C(\mathcal{A})}$. ♣ Preserves identities.

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$$\mathbf{Hmo}_{\text{ff}} \longrightarrow \mathbf{Ho} B_{\infty}$$

$$\mathcal{A} \longmapsto C(\mathcal{A})$$

$$\text{rep}_{\text{ff}}(\mathcal{A}, \mathcal{B}) \ni X \longmapsto \mathcal{C}(X) = \iota_{\mathcal{A}}^* (\iota_{\mathcal{B}}^*)^{-1}$$

Where the upper triangular matrix dg category $\mathcal{T}_X = \begin{pmatrix} \mathcal{A} & X \\ & \mathcal{B} \end{pmatrix}$.

Q2: Can we extend $\text{rep}_{\text{ff}}(\mathcal{A}, \mathcal{B})$ to all dg bimodules $\mathcal{D}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$?

An inspiring work of Keller

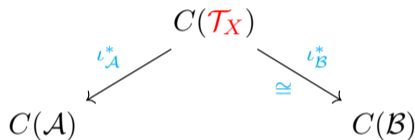
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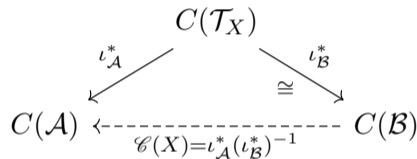
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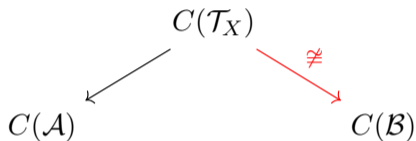
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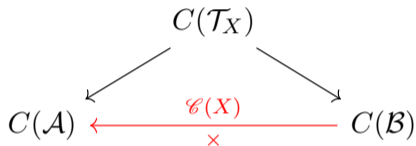
From functor to lax functor



When $X \in \text{rep}_{\text{ff}}(\mathcal{A}, \mathcal{B}), Y \in \text{rep}_{\text{ff}}(\mathcal{B}, \mathcal{C})$:

♣ The composition of spans should be given by **pullbacks**.

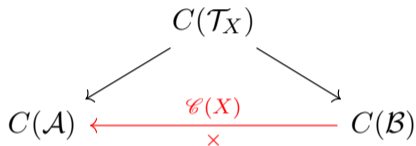
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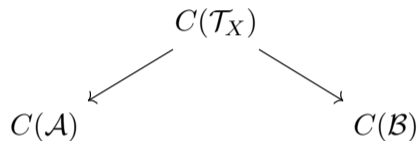
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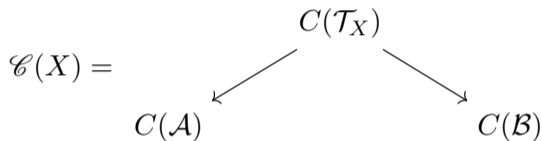
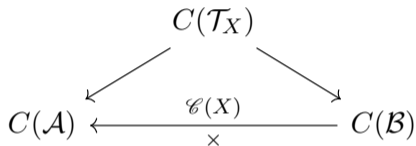
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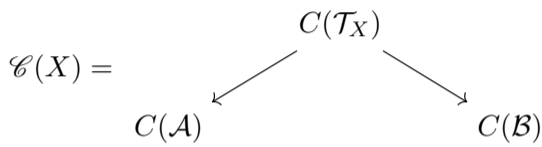
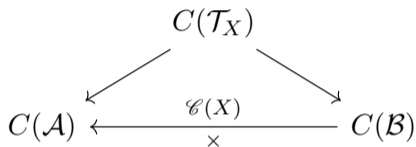
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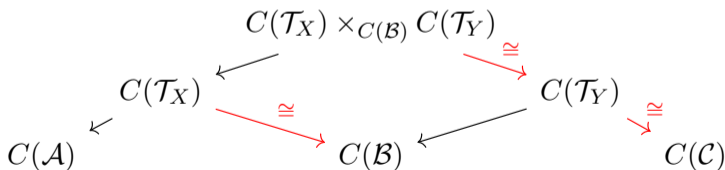
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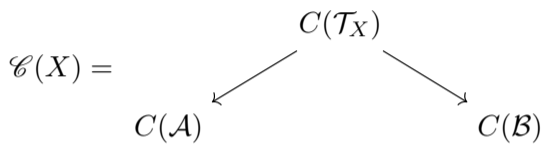
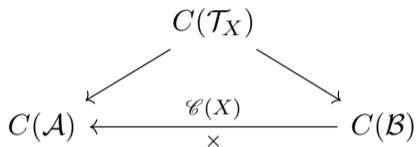


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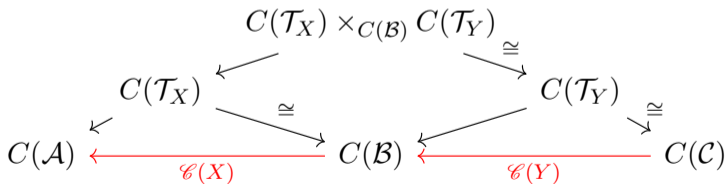


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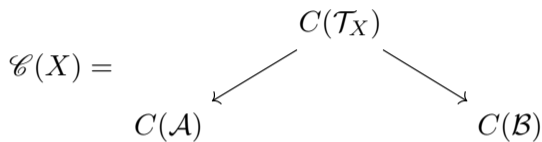
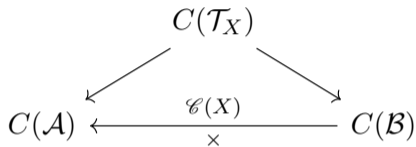


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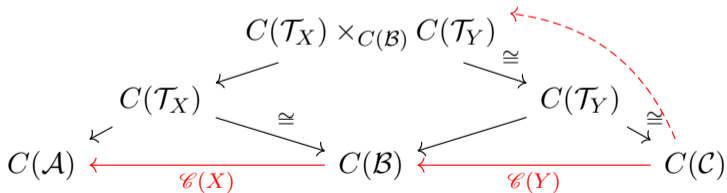


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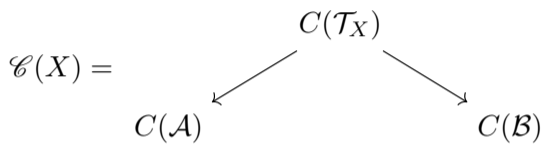
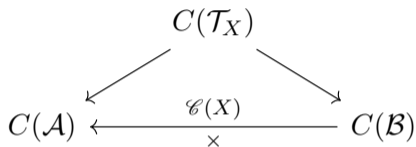


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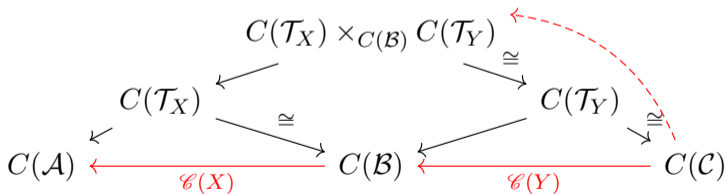


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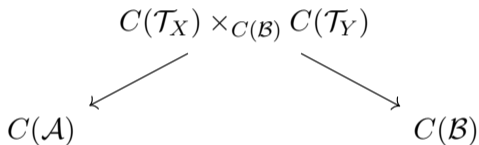


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From associativity to lax associativity constraints

The span $\mathcal{C}(X) \circ \mathcal{C}(Y)$:

The span $\mathcal{C}(X \otimes_B^L Y)$:

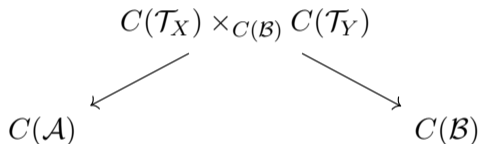


Problem: In general, $C(\mathcal{T}_X) \times_{C(\mathcal{B})} C(\mathcal{T}_Y) \not\cong C(\mathcal{T}_{X \otimes_B^L Y})$, so $\mathcal{C}(X) \circ \mathcal{C}(Y) \neq \mathcal{C}(X \otimes_B^L Y)$.

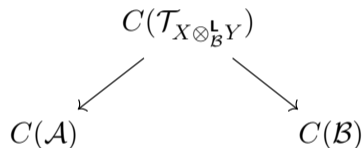
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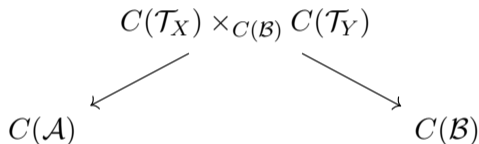


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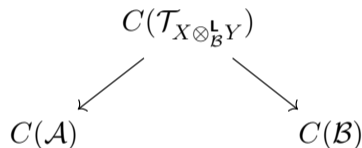
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The span $\mathcal{C}(X \otimes_B^L Y)$:

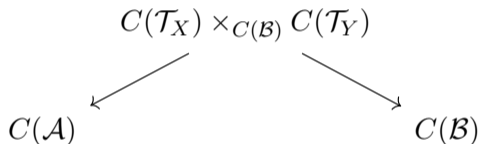


Problem: In general, $C(\mathcal{T}_X) \times_{C(\mathcal{B})} C(\mathcal{T}_Y) \not\cong C(\mathcal{T}_{X \otimes_B^L Y})$, so $\mathcal{C}(X) \circ \mathcal{C}(Y) \neq \mathcal{C}(X \otimes_B^L Y)$.

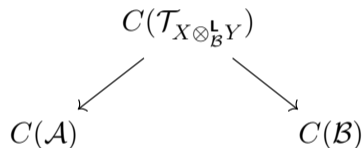
► This construction **does not preserve compositions**, i.e. this construction is a **lax functor**.

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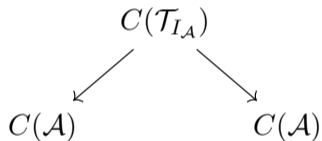


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From unity to lax unity constraints

The span $\mathcal{C}(I_{\mathcal{A}})$:



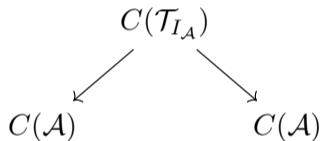
The span $\text{id}_{C(\mathcal{A})}$:

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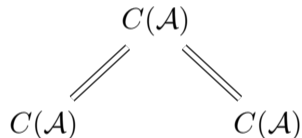
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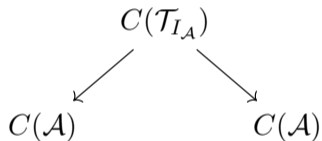


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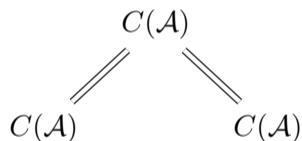
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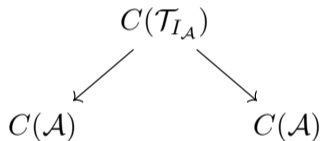


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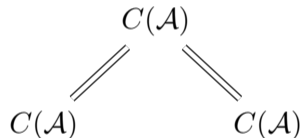
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From functor to lax functor

$$\begin{array}{ccc}
 \mathbf{Hmo}_{\text{ff}} & \longrightarrow & \text{Ho } B_{\infty} \\
 \mathcal{A} & \longmapsto & C(\mathcal{A}) \\
 \text{rep}_{\text{ff}}(\mathcal{A}, \mathcal{B}) \ni X & \longmapsto & \mathcal{C}(X)
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Q2: Can we extend $\text{rep}_{\text{ff}}(\mathcal{A}, \mathcal{B})$ to all dg bimodules $\mathcal{D}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$?

A2: Yes, but we should use the language of **bicategory** and **lax functor**.

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Constructions of bicategories

From category to bicategory

A **category** consists of

- ▶ A class of objects.
- ▶ Hom **sets**.
- ▶ For each object, an **identity morphism**.
- ▶ **Composition functions** between Hom sets.

The above data satisfy:

- ♣ **Equalities** of associativity.
- ♣ **Equalities** of unity.

A **bicategory** consists of

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Construction of source bicategories

$$\begin{array}{ccc} \mathbf{Hmo}_{\text{ff}} & \longrightarrow & \text{Ho } B_{\infty} \\ \mathcal{A} & \longmapsto & C(\mathcal{A}) \\ \text{rep}_{\text{ff}}(\mathcal{A}, \mathcal{B}) \ni X & \longmapsto & \mathcal{C}(X) \end{array}$$

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The bicategory cdgCAT is defined by

- ▶ Objects are small dg categories cofibrant over k .
- ▶ 1-cells are dg bimodules.
- ▶ Identity 1-cell is $I_{\mathcal{A}}$.
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The bicategory $\text{cdgCAT}_{c,h}$ is defined by

- ▶ Objects are small dg categories cofibrant over k .
- ▶ 1-cells are cofibrant dg bimodules.
- ▶ Identity 1-cell is $\mathbf{p}I_{\mathcal{A}}$.
- ▶ 2-cells are quasi-isomorphisms.
- ▶ Composition functor is $- \otimes_{\mathcal{B}} -$.
- ▶ Unitors and associators are natural isomorphisms.

Lemma

*The bicategories cdgCAT and $\text{cdgCAT}_{c,h}$ are **biequivalent**.*

Consturction of source bicategories

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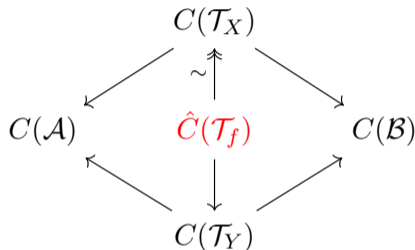
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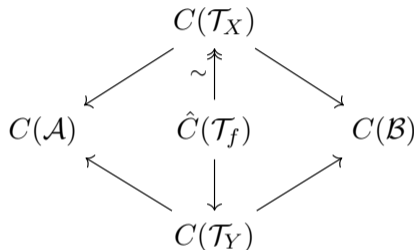
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Construction of target bicategories

Theorem

For each model category M , there is a bicategory $M\text{-span}^2$ given by

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Corollary

There are two bicategories $B_\infty\text{-span}^2$ and $\text{Ho } B_\infty\text{-span}^2$.

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Constructions of lax functors

Construction of lax functor

$$\begin{array}{l}
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Problem: What is the image $\mathcal{C}(f)$ of the morphism $f : X \rightarrow Y$?

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Property of upper triangular matrix dg category of $I_{\mathcal{A}}$

Lemma

There is a B_{∞} -quasi-isomorphism $\theta_{\mathcal{A}} : C(\mathcal{A}) \rightarrow C(\mathcal{T}_{I_{\mathcal{A}}})$.

The image of a morphism of dg bimodules

For each **quasi-isomorphism** $f : X \rightarrow Y$, it induces a **quasi-equivalence** $\mathcal{T}_f : \mathcal{T}_X \rightarrow \mathcal{T}_Y$.

♣ [Keller2003] The quasi-equivalence \mathcal{T}_f induces a dg \mathcal{T}_X - \mathcal{T}_Y -bimodule $X_{\mathcal{T}_f}$.

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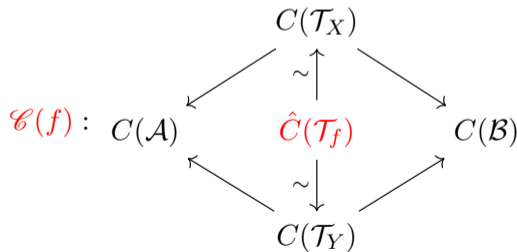
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Problem: What is the lax functoriality constraint $\mathcal{C}^2 : \mathcal{C}(Y) \circ \mathcal{C}(X) \Rightarrow \mathcal{C}(Y \circ X)$?

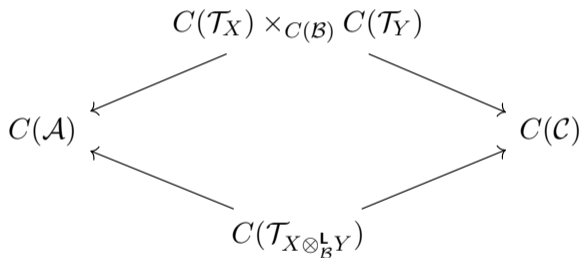
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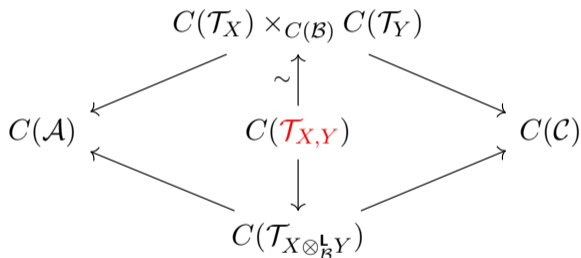
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where the upper triangular matrix dg category $\mathcal{T}_{X,Y} = \begin{pmatrix} \mathcal{A} & X & X \otimes_B Y \\ & \mathcal{B} & Y \\ & & \mathcal{C} \end{pmatrix}$.

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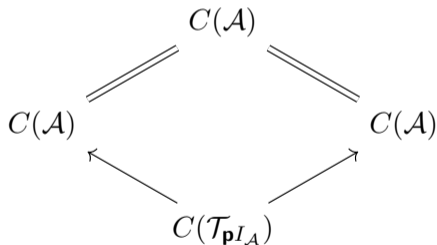
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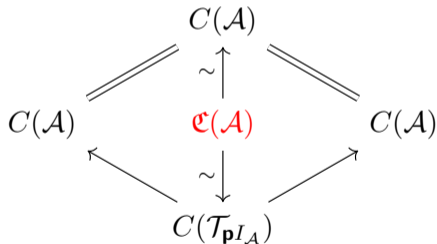
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 // & & // \\
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The lax functoriality of Hochschild cochain complex

Theorem

The above data $(\mathcal{C}, \mathcal{C}^2, \mathcal{C}^0) : \text{cdgCAT}_{c,h} \rightarrow B_\infty\text{-span}^2$ is a **lax functor**.

Furthermore, there is a **lax functor** of Hochschild cochain complex

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The functoriality of Hochschild cohomology

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The localization functor induces a *lax functor* $B_\infty\text{-span}_{\text{ra}}^2 \rightarrow \text{Ho } B_\infty\text{-span}^2$, where $B_\infty\text{-span}_{\text{ra}}^2$ is a sub-bicategory of $B_\infty\text{-span}^2$.

Corollary (Keller2006)

There is a *lax functor* of Hochschild cohomology $\mathcal{H}\mathcal{H} : \text{cdgCAT}_{\text{ff}} \rightarrow \text{Ho } B_\infty\text{-span}^2$, which induces the *functor*

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Comparing with Grady-Oren's work

Theorem (Grady-Oren 2021)

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$$\begin{array}{ccc}
 \mathbf{Hmo}_{\text{ff}} & \longrightarrow & \mathbf{Ho} B_{\infty} \\
 \mathcal{A} & \longmapsto & C(\mathcal{A}) \\
 \text{rep}_{\text{ff}}(\mathcal{A}, \mathcal{B}) \ni X & \longmapsto & \mathcal{C}(X)
 \end{array}$$

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 \text{cdgCAT}_{c,h} & \longrightarrow & B_{\infty}\text{-span}^2 \\
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 X \xrightarrow{f} Y & \longmapsto & \mathcal{C}(f)
 \end{array}$$

We extend the $\text{rep}_{\text{ff}}(\mathcal{A}, \mathcal{B})$ to all dg bimodules $\mathcal{D}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$,
 and show that there is a **lax functor** that generalizes the constructions.

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References

-  B. Keller, Deriving DG categories, *Ann. Sci. École Norm. Sup. (4)* 27 (1994), no. 1, 63–102.
-  B. Keller, Derived invariance of higher structures on the Hochschild complex, Available from <https://webusers.imj-prg.fr/~bernhard.keller/publ/dih.pdf>, 2003.
-  B. Keller, On differential graded categories, *International Congress of Mathematicians II*, European Mathematical Society, Zürich, 2006, 151–190.
-  R. Grady and G. Oren, Lax functors, cospans and the center construction, *Grad. J. Math.* 6 (2021), no. 2, 1–8.
-  N. Johnson and D. Yau, *2-dimensional categories*, Oxford University Press, Oxford, 2021.

Thanks!