The lax functoriality of Hochschild cochain complex

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Joint work with Yang Han

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Motivations

The functoriality of Hochschild chain complex/homology

- The Hochschild chain complex is a functor $C_* : k\text{-}\mathbf{alg} \longrightarrow \mathcal{C}k$

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd} \longmapsto \begin{tikzcd} C_*(A): \arrow[d, "C_*(f)"] & \cdots \arrow[r, "d_n"] & A^{\otimes n} \arrow[r, "d_{n-1}"] \arrow[d, "f^{\otimes n}"] & \cdots \arrow[r, "d_2"] & A^{\otimes 2} \arrow[r, "d_1"] \arrow[d, "f^{\otimes 2}"] & A \arrow[d, "f"] \\ C_*(B): & \cdots \arrow[r, "d_n"] & B^{\otimes n} \arrow[r, "d_{n-1}"] & \cdots \arrow[r, "d_2"] & B^{\otimes 2} \arrow[r, "d_1"] & B \end{tikzcd}$$

- The Hochschild homology is a functor

$$HH_* = H_* \circ C_* : k\text{-}\mathbf{alg} \longrightarrow \mathbf{Gr}k.$$

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B \arrow[ru, phantom, "\longmapsto"] & C_*(B) : & \cdots \arrow[r, "d_n"] & B^{\otimes n} \arrow[r, "d_{n-1}"] & \cdots \arrow[r, "d_2"] & B^{\otimes 2} \arrow[r, "d_1"] & B
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- The Hochschild homology is a functor

$$HH_* = H_* \circ C_* : k\text{-}\mathbf{alg} \longrightarrow \mathbf{Gr}k.$$

The Hochschild cochain complex/cohomology are not functors

The Hochschild cochain complex of k-algebras is not a functor.

The Hochschild cohomology of k-algebras is not a functor.

The center of k-algebras is not a functor.

Example

Let $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}, S = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$. Then $Z(A) \cong k, Z(S) = S \cong k^2$.

$$\begin{tikzcd} S \arrow[r, hook] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r, two heads] & S \end{tikzcd} \implies \begin{tikzcd} k^2 \arrow[r, hook] \arrow[rr, bend left, "\mathrm{id}"] & k \arrow[r, two heads] & k^2 \end{tikzcd}$$

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The lax functoriality of center

Theorem (Grady-Oren 2021)

The center of k-algebras is a lax functor.

Q1: Can we extend the lax functoriality of center to Hochschild cochain complex?

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An inspiring work of Keller

Theorem (Keller2003)

Let $\mathcal{A},\mathcal{B},\mathcal{C}$ *be dg categories cofibrant over* k*,*
and ${}_{\mathcal{A}}X_{\mathcal{B}}, {}_{\mathcal{B}}Y_{\mathcal{C}}$ *are cofibrant dg bimodules,*
such that $-\otimes^{\mathbf{L}}_{\mathcal{A}} X : \mathrm{per}\mathcal{A} \to \mathcal{D}\mathcal{B}$ *is fully faithful.*
Then there is a morphism $\mathscr{C}(X) : C(\mathcal{B}) \to C(\mathcal{A})$ *in* $\mathrm{Ho}\,B_\infty$*.*

- *Under more conditions,* $\mathscr{C}(X \otimes^{\mathbf{L}}_{\mathcal{B}} Y) = \mathscr{C}(X) \circ \mathscr{C}(Y)$. ♣ *Preserves compositions.*
- $\mathscr{C}(I_{\mathcal{A}}) = \mathrm{id}_{C(\mathcal{A})}$. ♣ *Preserves identities.*

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An inspiring work of Keller

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\begin{aligned}
\mathbf{Hmo}_{\mathrm{ff}} &\longrightarrow \mathrm{Ho}\, B_\infty \\
\mathcal{A} &\longmapsto C(\mathcal{A}) \\
\mathrm{rep}_{\mathrm{ff}}(\mathcal{A}, \mathcal{B}) \ni X &\longmapsto \mathscr{C}(X) = \iota_{\mathcal{A}}^*(\iota_{\mathcal{B}}^*)^{-1}
\end{aligned}
$$

Where the upper triangular matrix dg category $\mathcal{T}_X = \begin{pmatrix} \mathcal{A} & X \\ & \mathcal{B} \end{pmatrix}$.

Q2: Can we extend $\mathrm{rep}_{\mathrm{ff}}(\mathcal{A}, \mathcal{B})$ to all dg bimodules $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$?

An inspiring work of Keller

$$\begin{tikzcd}
\mathbf{Hmo}_{\mathrm{ff}} \arrow[r] & \operatorname{Ho} B_{\infty} \\
\mathcal{A} \arrow[r, mapsto] & C(\mathcal{A}) \\
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From funtor to lax functor

$$\begin{tikzcd}
& C(\mathcal{T}_X) \arrow[dl] \arrow[dr, "\ncong"] & \\
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\end{tikzcd}$$

When $X \in \mathrm{rep}_{\mathrm{ff}}(\mathcal{A}, \mathcal{B}), Y \in \mathrm{rep}_{\mathrm{ff}}(\mathcal{B}, \mathcal{C})$:

♣ The composition of spans should be given by pullbacks.

From funtor to lax functor

$$\begin{tikzcd}
 & C(\mathcal{T}_X) \arrow[ld] \arrow[rd] & \\
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& & C(\mathcal{T}_X) \times_{C(\mathcal{B})} C(\mathcal{T}_Y) \arrow[ld] \arrow[rd, red, "\cong"] & & \\
& C(\mathcal{T}_X) \arrow[ld] \arrow[rd, red, "\cong"] & & C(\mathcal{T}_Y) \arrow[ld] \arrow[rd, red, "\cong"] & \\
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From associativity to lax associativity constraints

The span $\mathscr{C}(X) \circ \mathscr{C}(Y)$:

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The span $\mathscr{C}(X \otimes^{\mathbf{L}}_{\mathcal{B}} Y)$:

Problem: In general, $C(\mathcal{T}_X) \times_{C(\mathcal{B})} C(\mathcal{T}_Y) \ncong C(\mathcal{T}_{X \otimes^{\mathbf{L}}_{\mathcal{B}} Y})$, so $\mathscr{C}(X) \circ \mathscr{C}(Y) \neq \mathscr{C}(X \otimes^{\mathbf{L}}_{\mathcal{B}} Y)$.

- ▶ This construction does not preserve compositions, i.e. this construction is a lax functor.

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Problem: In general, $C(\mathcal{T}_X)\times_{C(\mathcal{B})} C(\mathcal{T}_Y)\ncong C(\mathcal{T}_{X\otimes^{\mathsf{L}}_{\mathcal{B}} Y})$, so $\mathscr{C}(X)\circ\mathscr{C}(Y)\neq\mathscr{C}(X\otimes^{\mathsf{L}}_{\mathcal{B}} Y)$.

- This construction does not preserve compositions, i.e. this construction is a lax functor.

From associativity to lax associativity constraints

The span $\mathscr{C}(X) \circ \mathscr{C}(Y)$:

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 & C(\mathcal{T}_X) \times_{C(\mathcal{B})} C(\mathcal{T}_Y) \arrow[dl] \arrow[dr] & \\
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From unity to lax unity constraints

The span $\mathscr{C}(I_{\mathcal{A}})$:

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& C(\mathcal{T}_{I_{\mathcal{A}}}) \arrow[dl] \arrow[dr] & \\
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The span $\mathrm{id}_{C(\mathcal{A})}$:

Problem: In general, $C(\mathcal{T}_{I_{\mathcal{A}}}) \neq C(\mathcal{A})$, so $\mathscr{C}(I_{\mathcal{A}}) \neq \mathrm{id}_{C(\mathcal{A})}$.

- This construction does not preserve identities, i.e. this construction is a lax functor.

From unity to lax unity constraints

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The span $\mathrm{id}_{C(\mathcal{A})}$:

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From functor to lax functor

$$\begin{aligned} \mathbf{Hmo}_{\mathrm{ff}} &\longrightarrow \mathrm{Ho}\,B_\infty \\ \mathcal{A} &\longmapsto C(\mathcal{A}) \\ \mathrm{rep}_{\mathrm{ff}}(\mathcal{A},\mathcal{B}) \ni X &\longmapsto \mathscr{C}(X) \end{aligned}$$

$$\begin{aligned} \mathbf{Hmo}_{\mathrm{ff}} &\longrightarrow \mathrm{Ho}\,B_\infty \\ \mathcal{A} &\longmapsto C(\mathcal{A}) \\ \mathcal{D}(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}) \ni X &\longmapsto \mathscr{C}(X) \\ X \xrightarrow{f} Y &\longmapsto \mathscr{C}(f) =? \end{aligned}$$

Q2: Can we extend $\mathrm{rep}_{\mathrm{ff}}(\mathcal{A},\mathcal{B})$ to all dg bimodules $\mathcal{D}(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B})$?

A2: Yes, but we should use the language of bicategory and lax functor.

From functor to lax functor

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Constructions of bicategories

From category to bicategory

A category consists of

- A class of objects.
- Hom sets.
- For each object, an identity morphism.
- Composition functions between Hom sets.

The above data satisfy:

- ♣ Equalities of associativity.
- ♣ Equalities of unity.

A bicategory consists of

- A class of objects.
- Hom categories.
- For each object, an identity 1-cell.
- Composition functors between Hom categories.

The above data satisfy:

- ♣ Natural isomorphisms of associativity.
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From functor to lax functor

A functor F is consist of

- A function on objects.
- Functions between Hom sets.

The above data satisfy

- $F(g) \circ F(f) = F(g \circ f)$
- $\mathrm{id}_{FX} = F(\mathrm{id}_X)$

A lax functor (F, F^2, F^0) is consist of

- A function on objects.
- Functors between Hom categories.
- Lax functoriality constraint
  $F^2 : F(g) \circ F(f) \Rightarrow F(g \circ f)$
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Consturction of source bicategories

$$\begin{aligned} \mathbf{Hmo}_{\mathrm{ff}} &\longrightarrow \mathrm{Ho}\, B_\infty \\ \mathcal{A} &\longmapsto C(\mathcal{A}) \\ \mathrm{rep}_{\mathrm{ff}}(\mathcal{A},\mathcal{B}) \ni X &\longmapsto \mathscr{C}(X) \end{aligned}$$

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The bicategory $\mathrm{cdg}\mathcal{CAT}$ is defined by

- Objects are small dg categories cofibrant over k.
- 1-cells are dg bimodules.
- Identity 1-cell is $I_{\mathcal{A}}$.
- 2-cells are isomorphisms.
- Composition functor is $- \otimes^{\mathbf{L}}_{\mathcal{B}} -$.
- Unitors and associators are natural isomorphisms.

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- Objects are small dg categories cofibrant over k.
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Lemma

The bicategories $\mathrm{cdg}\mathcal{CAT}$ and $\mathrm{cdgCAT}_{\mathrm{c,h}}$ are biequivalent.

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Lemma

The bicategories $\mathrm{cdg}\mathcal{CAT}$ *and* $\mathrm{cdgCAT}_{\mathrm{c,h}}$ *are biequivalent.*

Consturction of source bicategories

The bicategory $\mathrm{cdg}\mathcal{CAT}$ is defined by

- Objects are small dg categories cofibrant over k.
- 1-cells are dg bimodules.
- Identity 1-cell is $I_{\mathcal{A}}$.
- 2-cells are isomorphisms.
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The bicategory $\mathrm{cdgCAT}_{\mathrm{c,h}}$ is defined by

- Objects are small dg categories cofibrant over k.
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- Identity 1-cell is $\mathbf{p}I_{\mathcal{A}}$.
- 2-cells are quasi-isomorphisms.
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Consturction of target bicategory

$$\begin{aligned} \mathrm{cdg}\mathcal{CAT} &\longrightarrow B_\infty\text{- span}^2 \\ \mathcal{A} &\longmapsto C(\mathcal{A}) \\ \mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}) \ni X &\longmapsto \mathscr{C}(X) \\ X \xrightarrow{f} Y &\longmapsto \mathscr{C}(f)? \end{aligned}$$

$$C(\mathcal{A}) \twoheadleftarrow C(\mathcal{T}_X) \twoheadrightarrow C(\mathcal{B})$$

$$C(\mathcal{A}) = C(\mathcal{A}) = C(\mathcal{A})$$

The bicategory $B_\infty\text{- span}^2$ is defined by

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- 2-cells are some equivalence classes of special spans of spans.
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Consturction of target bicategories

Theorem

For each model category M*, there is a bicategory* $\mathsf{M}\text{-}\mathrm{span}^2$ *given by*

- *Objects are objects of* M*.*
- *1-cells are spans of* M *with two fibrations.*
- *Identity 1-cell is* $A = A = A$*.*
- *2-cells are some equivalent classes of special spans of spans.*
- *Composition functor is given by pullbacks.*
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Corollary

There are two bicategories $B_\infty\text{-}\mathrm{span}^2$ *and* $\mathrm{Ho}\, B_\infty\text{-}\mathrm{span}^2$*.*

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Constructions of lax functors

Construction of lax functor

$$\begin{aligned} (\mathscr{C},\mathscr{C}^2,\mathscr{C}^0):\mathrm{cdgCAT}_{\mathrm{c,h}} &\longrightarrow B_\infty\text{-}\mathrm{span}^2\\ \mathcal{A} &\longmapsto C(\mathcal{A})\\ X &\longmapsto \mathscr{C}(X)\colon C(\mathcal{A}) \leftarrow C(\mathcal{T}_X)\to C(\mathcal{B})\\ X\xrightarrow{f}Y &\longmapsto \mathscr{C}(f)=?\end{aligned}$$

Problem: What is the image $\mathscr{C}(f)$ of the morphism $f : X \to Y$?

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X \arrow[r, mapsto] & \mathscr{C}(X) \colon C(\mathcal{A}) \leftarrow C(\mathcal{T}_X) \rightarrow C(\mathcal{B}) \\
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Property of upper triangular matrix dg category of $I_{\mathcal{A}}$

Lemma

There is a B_∞-quasi-isomorphism $\theta_{\mathcal{A}} : C(\mathcal{A}) \to C(\mathcal{T}_{I_{\mathcal{A}}})$.

The image of a morphism of dg bimodules

For each quasi-isomorphism $f : X \to Y$, it induces a quasi-equivalence $\mathcal{T}_f : \mathcal{T}_X \to \mathcal{T}_Y$.

♣ [Keller2003] The quasi-equivalence $\mathcal{T}_f$ induces a dg $\mathcal{T}_X$-$\mathcal{T}_Y$-bimodule $X_{\mathcal{T}_f}$.

$$C(\mathcal{T}_X) \xleftarrow{\sim} C(\mathcal{T}_{X_{\mathcal{T}_f}}) \xrightarrow{\sim} C(\mathcal{T}_Y)$$

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\hat{C}(\mathcal{T}_f) \arrow[r] \arrow[d] & C(\mathcal{A}) \times C(\mathcal{B}) \arrow[d, "\theta_{\mathcal{A}} \times \theta_{\mathcal{B}}"] \\
C(\mathcal{T}_{X_{\mathcal{T}_f}}) \arrow[r] & C(\mathcal{T}_{I_{\mathcal{A}}}) \times C(\mathcal{T}_{I_{\mathcal{B}}})
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\end{tikzcd}$$

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\mathscr{C}(f) : \quad C(\mathcal{A}) & \hat{C}(\mathcal{T}_f) \arrow[u, "\sim"'] \arrow[d, "\sim"'] & C(\mathcal{B}) \\
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\end{tikzcd}$$

Construction of lax functor

$$\begin{aligned} (\mathscr{C}, \mathscr{C}^2, \mathscr{C}^0) : \mathrm{cdgCAT}_{\mathrm{c,h}} &\longrightarrow B_\infty\text{-}\mathrm{span}^2 \\ \mathcal{A} &\longmapsto C(\mathcal{A}) \\ X &\longmapsto \mathscr{C}(X) \colon C(\mathcal{A}) \leftarrow C(\mathcal{T}_X) \rightarrow C(\mathcal{B}) \\ X \xrightarrow{f} Y &\longmapsto \mathscr{C}(f) \end{aligned}$$

Problem: What is the lax functoriality constraint $\mathscr{C}^2 : \mathscr{C}(Y) \circ \mathscr{C}(X) \Rightarrow \mathscr{C}(Y \circ X)$?

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Construction of lax functoriality constraint

A natural transformation $\mathscr{C}^2 : \mathscr{C}(Y) \circ \mathscr{C}(X) \Rightarrow \mathscr{C}(Y \circ X)$ is the equivalence class of span of spans

$$\begin{tikzcd}
 & C(\mathcal{T}_X) \times_{C(\mathcal{B})} C(\mathcal{T}_Y) \arrow[dl] \arrow[dr] & \\
C(\mathcal{A}) & & C(\mathcal{C}) \\
 & C(\mathcal{T}_{X \otimes^{\mathsf{L}}_{\mathcal{B}} Y}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where the upper triangular matrix dg category $\mathcal{T}_{X,Y} = \begin{pmatrix} \mathcal{A} & X & X \otimes_{\mathcal{B}} Y \\ & \mathcal{B} & Y \\ & & \mathcal{C} \end{pmatrix}$.

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\mathfrak{C}(\mathcal{A}) \arrow[r] \arrow[d] & C(\mathcal{A}) \arrow[d, "\theta_\mathcal{A}"] \\
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The lax functoriality of Hochschild cochian complex

Theorem

The above data $(\mathscr{C}, \mathscr{C}^2, \mathscr{C}^0) : \mathrm{cdgCAT}_{\mathrm{c,h}} \to B_\infty\text{-}\mathrm{span}^2$ *is a lax functor.*

Furthermore, there is a lax functor of Hochschild cochain complex

$$\mathscr{C} : \mathrm{cdg}\mathcal{CAT} \to B_\infty\text{-}\mathrm{span}^2 .$$

The lax functoriality of Hochschild cochian complex

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The functoriality of Hochschild cohomology

Lemma

The localization functor induces a lax functor $B_\infty\text{-}\mathrm{span}^2_{\mathrm{ra}} \to \mathrm{Ho}\, B_\infty\text{-}\mathrm{span}^2$, *where* $B_\infty\text{-}\mathrm{span}^2_{\mathrm{ra}}$ *is a sub-bicategory of* $B_\infty\text{-}\mathrm{span}^2$.

Corollary (Keller2006)

There is a lax functor of Hochschild cohomology $\mathscr{HH} : \mathrm{cdg}\mathcal{CAT}_{\mathrm{ff}} \to \mathrm{Ho}\, B_\infty\text{-}\mathrm{span}^2$, *which induces the functor*

$$\mathscr{C} : \mathbf{Hmo}_{\mathrm{ff}} \to \mathrm{Ho}\, B_\infty.$$

The functoriality of Hochschild cohomology

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Comparing with Grady-Oren's work

Theorem (Grady-Oren 2021)

The center of k-algebras is a lax functor.

We extend the lax functoriality of center of k-algebras to the lax functoriality of Hochschild cochain complex of dg categories.

Theorem

The Hochschild cochain complex of dg categories is a lax functor.

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Comparing with Keller's work

$$\begin{tikzcd}
\mathbf{Hmo}_{\mathrm{ff}} \arrow[r] & \mathrm{Ho}\, B_\infty \\
\mathcal{A} \arrow[r, mapsto] & C(\mathcal{A}) \\
\mathrm{rep}_{\mathrm{ff}}(\mathcal{A},\mathcal{B}) \ni X \arrow[r, mapsto] & \mathscr{C}(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{cdgCAT}_{\mathrm{c,h}} \arrow[r] & B_\infty\text{-}\mathrm{span}^2 \\
\mathcal{A} \arrow[r, mapsto] & C(\mathcal{A}) \\
\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}) \ni X \arrow[r, mapsto] & \mathscr{C}(X) \\
X \xrightarrow{f} Y \arrow[r, mapsto] & \mathscr{C}(f)
\end{tikzcd}$$

We extend the $\mathrm{rep}_{\mathrm{ff}}(\mathcal{A},\mathcal{B})$ to all dg bimodules $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$, and show that there is a lax functor that generalizes the constructions.

Comparing with Keller's work

$$\begin{tikzcd}
\mathbf{Hmo}_{\mathrm{ff}} \arrow[r] & \mathrm{Ho}\, B_\infty \\
\mathcal{A} \arrow[r, mapsto] & C(\mathcal{A}) \\
\mathrm{rep}_{\mathrm{ff}}(\mathcal{A}, \mathcal{B}) \ni X \arrow[r, mapsto] & \mathscr{C}(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{cdgCAT}_{\mathrm{c,h}} \arrow[r] & B_\infty\text{-}\,\mathrm{span}^2 \\
\mathcal{A} \arrow[r, mapsto] & C(\mathcal{A}) \\
\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}) \ni X \arrow[r, mapsto] & \mathscr{C}(X) \\
X \xrightarrow{f} Y \arrow[r, mapsto] & \mathscr{C}(f)
\end{tikzcd}$$

We extend the $\mathrm{rep}_{\mathrm{ff}}(\mathcal{A}, \mathcal{B})$ to all dg bimodules $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$,
and show that there is a lax functor that generalizes the constructions.

Comparing with Keller's work

$$\begin{tikzcd}
\mathbf{Hmo}_{\mathrm{ff}} \arrow[r] & \mathrm{Ho}\, B_\infty \\
\mathcal{A} \arrow[r, mapsto] & C(\mathcal{A}) \\
\mathrm{rep}_{\mathrm{ff}}(\mathcal{A}, \mathcal{B}) \ni X \arrow[r, mapsto] & \mathscr{C}(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{cdgCAT}_{\mathrm{c,h}} \arrow[r] & B_\infty\text{-}\mathrm{span}^2 \\
\mathcal{A} \arrow[r, mapsto] & C(\mathcal{A}) \\
\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}) \ni X \arrow[r, mapsto] & \mathscr{C}(X) \\
X \xrightarrow{f} Y \arrow[r, mapsto] & \mathscr{C}(f)
\end{tikzcd}$$

We extend the $\mathrm{rep}_{\mathrm{ff}}(\mathcal{A}, \mathcal{B})$ to all dg bimodules $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$, and show that there is a lax functor that generalizes the constructions.

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Thanks!