On the acyclic quantum cluster algebras with principle coefficients

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Background

Background

$$\begin{tikzcd}
\begin{array}{c} \mathit{Cluster\ algebra} \\ (\mathit{Fomin} - \mathit{Zelevinsky}) \end{array} \arrow[r] \arrow[d] & \begin{array}{c} \mathit{Quantum\ cluster\ algebra} \\ (\mathit{Berenstein} - \mathit{Zelevinsky}) \end{array} \arrow[d] \\
\begin{array}{c} \mathit{Generalized\ cluster\ algebra} \\ (\mathit{Chekhov} - \mathit{Shapiro}) \end{array} \arrow[r] & \mathit{Generalized\ Quantum\ cluster\ algebra}
\end{tikzcd}$$

Quantum Cluster Algebras

Now the quantum deformation of a cluster algebra is a $\mathbb{Q}(q)$-algebra obtained by making each cluster into a quasi-commuting family $\{X_1, \ldots, X_m\}$; this means that $X_i X_j = q^{\lambda_{ij}} X_j X_i$ for a skew-symmetric integer $m \times m$ matrix $\Lambda = (\lambda_{ij})$. In doing so, we have to modify the mutation process and the exchange relations so that all the adjacent quantum clusters will also be quasi-commuting. This imposes the compatibility relation between the quasi-commutation matrix Λ and the exchange matrix $\tilde{B}$. Any compatible matrix pair $(\Lambda, \tilde{B})$ gives rise to a well defined quantum cluster algebra.

Set Up

Write $[s,t] : \{s, s+1, \ldots, t-1, t\}$ for $s < t$.

A square integer matrix B is called skew-symmetrizable if there exists some integer diagonal matrix D with positive diagonal entries such that DB is skew-symmetric, D: the skew-symmetrizer of B.

Let m and n be two positive integers with $m \geq n$.

Let $\widetilde{B} = (b_{ij})$ be an $m \times n$ integer matrix with its upper $n \times n$ submatrix being skew-symmetrizable denoted by B called the principal part of $\widetilde{B}$.

We can choose an $m \times m$ skew-symmetric integer matrix Λ such that $\widetilde{B}^T \Lambda = \begin{bmatrix} D & \mathbf{0} \end{bmatrix}$ for some integer diagonal matrix D with positive diagonal entries.

Compatible Pair

The pair $(\widetilde{B}, \Lambda)$ is called a compatible pair, which is specified by the following data:

1. an $m \times n$ integer matrix $\widetilde{B}$ with the skew-symmetrizable principal part B and its skew-symmetrizer $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and $\widetilde{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_n \end{bmatrix}$ with $\mathbf{b}_j \in \mathbb{Z}^m$ for $j \in [1, n]$;
2. a skew-symmetric bilinear form $\Lambda : \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$ satisfying the compatibility condition with $\widetilde{B}$, i.e.,

$$\Lambda(\mathbf{b}_j, \mathbf{e}_i) = \delta_{ij} d_j \quad (i \in [1, m], j \in [1, n])$$

   where $\mathbf{e}_i$ is the i-th unit vector in $\mathbb{Z}^m$ for any $i \in [1, m]$.

Note that we can identify the bilinear form Λ with the skew-symmetric $m \times m$ matrix still denoted by $\Lambda = (\lambda_{ij})$ with $\lambda_{ij} := \Lambda(\mathbf{e}_i, \mathbf{e}_j)$.

Quantum Torus

Let q be a formal variable and denote $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ the ring of integer Laurent polynomials in the variable $q^{\frac{1}{2}}$.

The based quantum torus $\mathcal{T} = \mathcal{T}(\Lambda)$ is the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra with a distinguished $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis $\{X^{\mathbf{e}} : \mathbf{e} \in \mathbb{Z}^m\}$ and the multiplication given by

$$X^{\mathbf{e}} X^{\mathbf{f}} = q^{\frac{\Lambda(\mathbf{e},\mathbf{f})}{2}} X^{\mathbf{e}+\mathbf{f}} \quad (\mathbf{e}, \mathbf{f} \in \mathbb{Z}^m).$$

Denote by $x_i = X^{\mathbf{e}_i}$ for any $i \in [1, m]$, then the elements of x_i and their inverses generate $\mathcal{T}$ as a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra, subject to the quasi-commutative relations

$$x_i x_j = q^{\lambda_{ij}} x_j x_i$$

for $i, j \in [1, m]$. For any $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m$, define

$$X^{\mathbf{a}} := q^{\frac{1}{2}\sum_{l<k} a_k a_l \lambda_{kl}} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}.$$

Quantum Seed

Definition

With the above notations, a quantum seed is defined to be the triple $(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$, where the set $\widetilde{\mathbf{x}} = \{x_1, x_2, \ldots, x_m\}$ is the extended cluster, $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ is the cluster, elements x_i for $i \in [1, n]$ are called quantum cluster variables and elements x_i for $i \in [m+1, n]$ are called frozen variables.

Define the function

$$[x]_+ := \begin{cases} x, & \text{if} \quad x \geq 0; \\ 0, & \text{if} \quad x < 0. \end{cases}$$

Mutation

Definition

For $k \in [1, n]$, the mutation of a quantum seed $(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$ in the direction k is the quantum seed $\mu_k(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B}) := (\widetilde{\mathbf{x}}', \Lambda', \widetilde{B}')$, where

(1) the set $\widetilde{\mathbf{x}}' := (\widetilde{\mathbf{x}} - \{x_k\}) \cup \{x'_k\}$ with

$$x'_k = X^{-\mathbf{e}_k + [\mathbf{b}_k]_+} + X^{-\mathbf{e}_k + [-\mathbf{b}_k]_+}; \tag{1}$$

(2) the matrix $\widetilde{B}' := \mu_k(\widetilde{B})$ is defined by

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise}; \end{cases} \tag{2}$$

Mutation

Definition

(3) the skew-symmetric matrix $\Lambda' := \mu_k(\Lambda)$ is defined by

$$\lambda'_{ij} = \begin{cases} \lambda_{ij}, & \text{if } i, j \neq k; \\ -\lambda_{ij} + \sum_{t=1}^{m} [b_{ti}]_+ \lambda_{tj}, & \text{if } i = k, j \neq k. \end{cases} \tag{3}$$

Note that μ_k is an involution. Two quantum seeds are called mutation-equivalent if one can be obtained from another by a sequence of mutations. Denote the skew-field of fractions of $\mathcal{T}$ by $\mathcal{F}$ and

$$\mathbb{ZP} := \mathbb{Z}[q^{\pm\frac{1}{2}}][x_{n+1}^{\pm}, \ldots, x_m^{\pm}].$$

Quantum Cluster Algebras

Definition

Given an initial quantum seed $(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$, the quantum cluster algebra $\mathcal{A}(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by all cluster variables from all quantum seeds mutation-equivalent to $(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$.

Note that one can recover the classical cluster algebra by setting $q = 1$.

The directed graph associated to a quantum seed $(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$ is denoted by $\Gamma(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$ with vertices $[1, n]$ and the directed edges from i to j if $b_{ij} > 0$.

Definition

A quantum seed $(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$ is called acyclic if $\Gamma(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$ does not contain any oriented cycle. A quantum cluster algebra is called acyclic if it has an acyclic quantum seed.

Quantum Cluster Algebras

Definition (Berenstein-Zelevinsky)
A standard monomial in $x_1, x_1', \ldots, x_n, x_n'$ is an element of the form $x_1^{a_1} \cdots x_n^{a_n} (x_1')^{a_1'} \cdots (x_n')^{a_n'}$, where all exponents are non-negative integers, and $a_k a_k' = 0$ for $k \in [1, n]$.

Denote by $\mathcal{L}(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B}) := \mathbb{ZP}[x_1, x_1', \ldots x_n, x_n']$. The following theorems are quantum versions of the corresponding results in Berensrein-Fomin-Zelevinsky: Cluster Algebras. III..

Theorem (Berenstein-Zelevinsky)
The standard monomials in $x_1, x_1', \ldots, x_n, x_n'$ are linearly independent over $\mathbb{ZP}$ (i.e., they form a $\mathbb{ZP}$-basis of $\mathcal{L}(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$) if and only if B is acyclic.

Theorem (Berenstein-Zelevinsky)
The condition that a quantum seed $\left(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B}\right)$ is acyclic, is necessary and sufficient for the equality $\mathcal{L}(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B}) = \mathcal{A}\left(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B}\right)$.

The Quantum Projective Cluster Variables

Quantum Projective Cluster Variables

Up to simultaneously reordering of columns and rows, we can assume that the entries in the skew-symmetrizable matrix B satisfy $b_{ij} \geq 0$ for any $i > j$ which then defines a linear order $\triangleleft$ on $[1, n]$. In the following, let $\Sigma = (\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$ be an acyclic quantum seed of an acyclic quantum cluster algebra with principal coefficients.

Definition

For any $i \in [1, n]$, define a new acyclic quantum seed

$$\Sigma^{(i)} = (\widetilde{\mathbf{x}}^{(i)}, \Lambda^{(i)}, \widetilde{B}^{(i)}) := \mu_i \cdots \mu_2 \mu_1 (\widetilde{\mathbf{x}}, \Lambda, \widetilde{B}),$$

where $\widetilde{\mathbf{x}}^{(i)} = \{x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}, x_{n+1}^{(i)} \ldots, x_{2n}^{(i)}\}$. Thus we have $\Sigma^{(n)} = (\widetilde{\mathbf{x}}^{(n)}, \Lambda^{(n)}, \widetilde{B}^{(n)})$. The cluster $\mathbf{x}^{(n)} = \{x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}\}$ is called the quantum projective cluster, and each cluster variable in $\mathbf{x}^{(n)}$ is called a quantum projective cluster variable.

Quantum Cluster Algebras with Principle Coefficients

Note that the new quantum seed $\Sigma^{(i)}$ is obtained by applying a sequence of mutations on Σ corresponding to a sink sequence of the directed graph $\Gamma(\Sigma)$ and by the above definition we have $B^{(n)} = B$, $x_i^{(i)} = x_i^{(j)}$ for any $1 \leq i \leq j \leq n$ and $x_i^{(j)} = x_i$ for any $i \in [n+1, 2n]$ and $j \in [1, n]$.

Quantum Cluster Algebras with Principle Coefficients

It is straightforward to obtain that

$$\widetilde{B}^{(i)} = \begin{bmatrix}
0 & b_{12} & \cdots & b_{1i} & -b_{1\,i+1} & -b_{1\,i+2} & \cdots & -b_{1\,n-1} & -b_{1n} \\
b_{21} & 0 & \cdots & b_{2i} & -b_{2\,i+1} & -b_{2\,i+2} & \cdots & -b_{2\,n-1} & -b_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
b_{i1} & b_{i2} & \cdots & 0 & -b_{i\,i+1} & -b_{i\,i+2} & \cdots & -b_{i\,n-1} & -b_{in} \\
-b_{i+1\,1} & -b_{i+1\,2} & \cdots & -b_{i+1\,i} & 0 & b_{i+1\,i+2} & \cdots & b_{i+1\,n-1} & b_{i+1\,n} \\
-b_{i+2\,1} & -b_{i+2\,2} & \cdots & -b_{i+2\,i} & b_{i+2\,i+1} & 0 & \cdots & b_{i+2\,n-1} & b_{i+2\,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-b_{n-1\,1} & -b_{n-1\,2} & \cdots & -b_{n-1\,i} & b_{n-1\,i+1} & b_{n-1\,i+2} & \cdots & 0 & b_{n-1\,n} \\
-b_{n1} & -b_{n2} & \cdots & -b_{ni} & b_{n\,i+1} & b_{n\,i+2} & \cdots & b_{n\,n-1} & 0 \\
-1 & & & & & & & & \\
 & -1 & & & & & & & \\
 & & \ddots & & & & & & \\
 & & & -1 & & & & & \\
 & & & & 1 & & & & \\
 & & & & & 1 & & & \\
 & & & & & & \ddots & & \\
 & & & & & & & 1 & \\
 & & & & & & & & 1
\end{bmatrix},$$

Quantum Cluster Algebras with Principle Coefficients

$$\Lambda^{(i)} = \begin{bmatrix}
0 & \lambda_{12} & \cdots & \lambda_{1i} & -\lambda_{1\,i+1} & -\lambda_{1\,i+2} & \cdots & -\lambda_{1\,2n-1} & -\lambda_{1\,2n} \\
\lambda_{21} & 0 & \cdots & \lambda_{2i} & -\lambda_{2\,i+1} & -\lambda_{2\,i+2} & \cdots & -\lambda_{2\,2n-1} & -\lambda_{2\,2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_{i1} & \lambda_{i2} & \cdots & 0 & -\lambda_{i\,i+1} & -\lambda_{i\,i+2} & \cdots & -\lambda_{i\,2n-1} & -\lambda_{i\,2n} \\
-\lambda_{i+1\,1} & -\lambda_{i+1\,2} & \cdots & -\lambda_{i+1\,i} & 0 & \lambda_{i+1\,i+2} & \cdots & \lambda_{i+1\,2n-1} & \lambda_{i+1\,2n} \\
-\lambda_{i+2\,1} & -\lambda_{i+2\,2} & \cdots & -\lambda_{i+2\,i} & \lambda_{i+2\,i+1} & 0 & \cdots & \lambda_{i+2\,2n-1} & \lambda_{i+2\,2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\lambda_{2n-1\,1} & -\lambda_{2n-1\,2} & \cdots & -\lambda_{2n-1\,i} & \lambda_{2n-1\,i+1} & \lambda_{2n-1\,i+2} & \cdots & 0 & \lambda_{2n-1\,2n} \\
-\lambda_{2n\,1} & -\lambda_{2n\,2} & \cdots & -\lambda_{2n\,i} & \lambda_{2n\,i+1} & \lambda_{2n\,i+2} & \cdots & \lambda_{2n\,2n-1} & 0
\end{bmatrix}$$

Quantum Cluster Algebras with Principle Coefficients

Definition

Given an acyclic quantum seed Σ. The lower bound quantum cluster algebra $\mathcal{L}^{(n)}(\Sigma)$ is defined to be the algebra generated by all initial and quantum projective cluster variables over $\mathbb{ZP}$, i.e.,

$$\mathcal{L}^{(n)}(\Sigma) = \mathbb{ZP}[x_1, x_1^{(n)}, \ldots, x_n, x_n^{(n)}].$$

It is obvious that $\mathcal{L}^{(n)}(\Sigma)$ is a finitely generated subalgebra of the quantum cluster algebra $\mathcal{A}(\Sigma)$.

Quantum Cluster Algebras with Principle Coefficients

The following well-known result will be used later.

Lemma

If $xy = q^{-1}yx$, then

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q[k-1]_q \cdots [1]_q}$ *and* $[n]_q = \frac{q^n - 1}{q-1}$.

Quantum Cluster Algebras with Principle Coefficients

The notation $\prod\limits_{j\in[1,m]}^{\lhd}$ means that the product is taken in increasing order with respect to $\lhd$.

Lemma

For any $k \in [1,n]$, we have

$$\begin{aligned} & (X^{(k-1)})^{-\mathbf{e}_k+[\mathbf{b}_k^{(k-1)}]_+} \\ = \quad & q^{\frac{1}{2}\left(-\sum\limits_{t=1}^{k-1}\sum\limits_{j=t+1}^{k-1} b_{tk}b_{jk}\lambda_{tj}+\sum\limits_{j=1}^{k-1} b_{jk}\lambda_{jk}+\sum\limits_{j=k+1}^{n} b_{jk}\lambda_{kj}+\lambda_{k\,n+k}\right)} \\ & \cdot \prod_{j\in[1,k-1]}^{\lhd} (x_j^{(k-1)})^{-b_{jk}} \cdot x_k^{-1}(X^{(k-1)})^{\sum\limits_{j=k+1}^{n} b_{jk}\mathbf{e}_j+\mathbf{e}_{n+k}}. \end{aligned}$$

Quantum Cluster Algebras with Principle Coefficients

Proposition

The algebra $\mathbb{ZP}[x_1, x_1', \ldots, x_n, x_n']$ *is a* $\mathbb{ZP}$*-subalgebra of* $\mathcal{L}^{(n)}(\Sigma)$.

Theorem

If $\Sigma = (\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$ *is an acyclic quantum seed, then*

$$\mathcal{A}(\Sigma) = \mathcal{L}^{(n)}(\Sigma).$$

The Dual PBW-Bases

Dual PBW-bases

In this section, we first establish a class of formulas for acyclic quantum cluster algebras with principal coefficients, and then construct the dual PBW bases of these algebras.

Lemma

For any $k \in [1, n]$ *and* $l \in \mathbb{N}$*, we have*

$$(x_k^{(n)})^l x_k - x_k (x_k^{(n)})^l = f_{k,l}^{(k)},$$

where $f_{k,l}^{(k)} = g_{k,k,l} \prod_{j\in[k+1,2n]}^{\triangleleft} x_j^{b_{jk}} \cdot \prod_{j\in[1,k-1]}^{\triangleleft} (x_j^{(n)})^{-b_{jk}} \cdot (x_k^{(n)})^{l-1}$ *and* $g_{k,k,l} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$.

Dual PBW-bases

Lemma

For any $j \in [1, n]$ and $l \in \mathbb{N}$, we have

$$(x_j^{(n)})^l x_{j-1} - q^{-l\lambda_{j\,j-1}} x_{j-1} (x_j^{(n)})^l = f_{j-1,l}^{(j)},$$

where $f_{j-1,1}^{(j)} \in \mathbb{ZP}[x_1^{(n)}, \dots, x_{j-1}^{(n)}, x_j, \dots, x_n]$, and $f_{j-1,l}^{(j)} \in \mathbb{ZP}[x_1^{(n)}, \dots, x_{j-1}^{(n)}, x_j^{(n)}, x_j, \dots, x_n]$ for any $l \geq 2$.

Dual PBW-bases

Lemma

For any $j \in [1, n]$ *and* $k < j$, *we have*

$$x_j^{(n)} x_k - q^{-\lambda_{jk}} x_k x_j^{(n)} = f_{k,1}^{(j)},$$

where $f_{k,1}^{(j)} = M_{k,1}^{(j)} (X^{(j-1)})^{\sum_{v=j+1}^{n} b_{vj}\mathbf{e}_v + \mathbf{e}_{n+j}}$ *and*

$$\begin{aligned} M_{k,1}^{(j)} \;&=\; \sum_{t=k}^{j-1} \Big(q^{M_j - \lambda_{jk} + \lambda_{n+j\,k} + \sum_{v=j+1}^{n} b_{vj}\lambda_{vk} + \sum_{r=t+1}^{j-1} b_{rj}\lambda_{rk}} \prod_{i\in[1,t-1]}^{\triangleleft} (x_i^{(n)})^{-b_{ij}} \cdot f_{k,-b_{tj}}^{(t)} \\ &\quad \cdot \prod_{i\in[t+1,j-1]}^{\triangleleft} (x_i^{(n)})^{-b_{ij}} \cdot x_j^{-1}\Big) \in \mathbb{ZP}[x_1^{(n)}, \ldots, x_{j-1}^{(n)}, x_{k+1}, \ldots, x_n], \end{aligned}$$

with $f_{k,-b_{tj}}^{(t)} := \sum_{i=1}^{-b_{tj}} q^{-(i-1)\lambda_{tk}} (x_t^{(n)})^{-b_{tj}-i} f_{k,1}^{(t)} (x_t^{(n)})^{i-1}$.

Dual PBW-bases

By above Lemma,

Corollary

For any $j \in [1, n]$ *and* $k < j$*, we have*

$$f_{k,1}^{(j)} \in \mathbb{ZP}[x_1^{(n)}, \ldots, x_{j-1}^{(n)}, x_{k+1}, \ldots, x_n].$$

Remark
The above result is in the same spirit described as Levendorskii-Soibelman straightening law.

Lemma

For any $j, k \in [1, n]$*,* $k < j$ *and* $l \in \mathbb{N}$*, we have*

$$x_j^{(n)} x_k^l - q^{-l\lambda_{jk}} x_k^l x_j^{(n)} = g_{k,l}^{(j)},$$

where $g_{k,l}^{(j)} \in \mathbb{ZP}[x_1^{(n)}, \ldots, x_{j-1}^{(n)}, x_k, \ldots, x_n]$.

Dual PBW-bases

Definition

A monomial $x_1^{a_1} \cdots x_n^{a_n} (x_1^{(n)})^{a_1^{(n)}} \cdots (x_n^{(n)})^{a_n^{(n)}}$ is called a projective standard if all exponents are non-negative integers and $a_k a_k^{(n)} = 0$ for all $k \in [1, n]$.

Dual PBW-bases

Proposition

For any $j, k \in [1, n], a_k, a_k^{(n)} \in \mathbb{N}$ such that $a_k a_k^{(n)} = 0$, the following items are all $\mathbb{ZP}$-linear combinations of projective standard monomials:

1. $x_j^{(n)} \prod_{k\in[1,n]}^{\triangleleft} x_k^{a_k} \cdot \prod_{k\in[1,n]}^{\triangleleft} (x_k^{(n)})^{a_k^{(n)}}$;
2. $\prod_{k\in[1,n]}^{\triangleleft} x_k^{a_k} \cdot \prod_{k\in[1,n]}^{\triangleleft} (x_k^{(n)})^{a_k^{(n)}} \cdot x_j^{(n)}$;
3. $x_j \prod_{k\in[1,n]}^{\triangleleft} x_k^{a_k} \cdot \prod_{k\in[1,n]}^{\triangleleft} (x_k^{(n)})^{a_k^{(n)}}$;
4. $\prod_{k\in[1,n]}^{\triangleleft} x_k^{a_k} \cdot \prod_{k\in[1,n]}^{\triangleleft} (x_k^{(n)})^{a_k^{(n)}} \cdot x_j$.

Dual PBW-bases

For any $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, we denote

$$\mathbf{x}^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}.$$

Let $\prec$ denote the lexicographic order on $\mathbb{Z}^n$, i.e., for any two vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{a}' = \left(a'_1, a'_2, \ldots, a'_n\right) \in \mathbb{Z}^n$ satisfy that $\mathbf{a} \prec \mathbf{a}'$ if and only if there exists $k \in [1, n]$ such that $a_k < a'_k$ and $a_i = a'_i$ for all $i \in [1, k-1]$. This order induces the lexicographic order on the Laurent monomials as

$$\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{a}'} \quad \text{if} \quad \mathbf{a} \prec \mathbf{a}'.$$

Definition

Let $Y = g_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} + \sum_{y} g_{\mathbf{a}_y} \mathbf{x}^{\mathbf{a}_y}$ for nonzero elements $g_{\mathbf{a}}, g_{\mathbf{a}_y} \in \mathbb{Z}\mathbb{P}$. We call $g_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ the first Laurent monomial of Y if $\mathbf{a}_y \prec \mathbf{a}$ for any y in some index set.

Dual PBW-bases

Theorem

Let $\Sigma = (\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$ *be an acyclic quantum seed. Then the projective standard monomials in* $x_1, x_1^{(n)}, \ldots, x_n, x_n^{(n)}$ *form a* $\mathbb{ZP}$*-basis of* $\mathcal{A}(\Sigma)$*.*

Remark

1. If we set $q = 1$ and B skew-symmtric, one can obtain a $\mathbb{ZP}$-basis of the classical acyclic cluster algebra which is proved in [Baur–Nasr-Isfahani: Cluster algebras generated by projective cluster variables, 2023].
2. If B is skew-symmtric, this basis should agree with an associated dual PBW basis in the language of [Kimura–Qin: Graded quiver varieties, quantum cluster algebras and dual canonical basis. 2014]. This is the reason why we call the basis in the above Theorem the dual PBW basis.

An Example

An Example

Example

Consider the acyclic quantum seed $(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$ as follows:

$$\widetilde{B} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Lambda = \begin{bmatrix} 0 & -1 & -1 & 1 & 2 & -2 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 \\ -2 & 0 & -1 & 1 & 0 & -2 \\ 2 & -1 & 0 & 1 & 2 & 0 \end{bmatrix}.$$

We have

$$\begin{aligned} x_1' &= q^{-\frac{1}{2}} x_1^{-1} x_2 x_3 x_4 + x_1^{-1}, \\ x_2' &= q^{-1} x_2^{-1} x_3^2 x_5 + q^{-\frac{1}{2}} x_1 x_2^{-1}, \\ x_3' &= x_3^{-1} x_6 + q^{\frac{1}{2}} x_1 x_2^2 x_3^{-1}. \end{aligned}$$

An Example

Example

By mutating the seed $(\widetilde{\mathbf{x}}, \Lambda, \widetilde{B})$ in the direction 1, we have

$$x_1^{(1)} = x_1' = q^{-\frac{1}{2}} x_1^{-1} x_2 x_3 x_4 + x_1^{-1},$$

$$\widetilde{B}^{(1)} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -2 \\ -1 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \Lambda^{(1)} = \begin{bmatrix} 0 & 1 & 1 & -1 & -2 & 2 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 \\ 2 & 0 & -1 & 1 & 0 & -2 \\ -2 & -1 & 0 & 1 & 2 & 0 \end{bmatrix}.$$

An Example

Example

By mutating the seed $(\widetilde{\mathbf{x}}^{(1)}, \Lambda^{(1)}, \widetilde{B}^{(1)})$ in the direction 2, we have

$$x_2^{(2)} = q^{-1} x_1^{-1} x_3^3 x_4 x_5 + q^{-\frac{1}{2}} x_1^{-1} x_2^{-1} x_3^2 x_5 + x_2^{-1}.$$

Multiplying both sides of the above equation from left by x_1, we have

$$x_1 x_2^{(2)} = q^{-1} x_3^3 x_4 x_5 + q^{\frac{1}{2}} x_2',$$

$$\widetilde{B}^{(2)} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ -1 & -2 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \Lambda^{(2)} = \begin{bmatrix} 0 & -1 & 1 & -1 & -2 & 2 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 \\ 2 & 0 & -1 & 1 & 0 & -2 \\ -2 & 1 & 0 & 1 & 2 & 0 \end{bmatrix}.$$

An Example

Example

By mutating the seed $(\widetilde{\mathbf{x}}^{(2)}, \Lambda^{(2)}, \widetilde{B}^{(2)})$ in the direction 3, we have

$$\begin{aligned} & x_3^{(3)} \\ = \; & q^{-1}x_1^{-1}(x_1^{(1)})^2x_2^{-1}x_3^4x_4x_5^2x_6 + (q^{\frac{1}{2}} + q^{\frac{3}{2}})x_1^{-1}x_1^{(1)}x_2^{-1}x_3^2x_4x_5x_6 \\ & +qx_1^{-1}x_2^{-1}x_4x_6 + q^{\frac{3}{2}}x_1^{-1}(x_1^{(1)})^2x_2^{-2}x_3^3x_5^2x_6 \\ & +(q^2 + q^3)x_1^{-1}x_1^{(1)}x_2^{-2}x_3x_5x_6 + q^{\frac{3}{2}}x_1^{-1}x_2^{-2}x_3^{-1}x_6 + x_3^{-1}. \end{aligned}$$

An Example

Example

Multiplying both sides of the above equality from left by $x_1 x_2^2$, we have

$$\begin{aligned}
& x_1 x_2^2 x_3^{(3)} \\
= \; & q^{-7}(x_1^{(1)})^2 x_2 x_3^4 x_4 x_5^2 x_6 + (q^{-\frac{7}{2}} + q^{-\frac{5}{2}}) x_1^{(1)} x_2 x_3^2 x_4 x_5 x_6 \\
& + q^{-1} x_2 x_4 x_6 + q^{-\frac{9}{2}}(x_1^{(1)})^2 x_3^3 x_5^2 x_6 + (q^{-2} + q^{-1}) x_1^{(1)} x_3 x_5 x_6 + q^{-\frac{1}{2}} x_3'
\end{aligned}$$

An Example

Example

Thus x_1', x_2' and $x_3' \in \mathbb{ZP}[x_1, x_1^{(3)}, x_2, x_2^{(3)}, x_3, x_3^{(3)}]$. Note that

$$\begin{aligned}
& x_1^{(3)}x_1 - x_1x_1^{(3)} = q^{-\frac{1}{2}}(q-1)x_2x_3x_4, \\
& x_2^{(3)}x_2 - x_2x_2^{(3)} = q^{-\frac{1}{2}}(1-q^{-1})x_1^{(3)}x_3^2x_5, \\
& x_3^{(3)}x_3 - x_3x_3^{(3)} = q^{\frac{1}{2}}(q-1)x_1^{(3)}(x_2^{(3)})^2x_6, \\
& x_2^{(3)}x_1 - q^{-1}x_1x_2^{(3)} = q^{-2}(q-1)x_3^3x_4x_5, \\
& x_3^{(3)}x_2 - x_2x_3^{(3)} = 2q^{-3}(1-q^{-1})x_3x_5x_6(x_1^{(3)})^2x_2^{(3)}, \\
& x_3^{(3)}x_1 - q^{-1}x_1x_3^{(3)} \\
= \; & q^{-\frac{1}{2}}(q-1)x_3^2x_4x_5x_6x_1^{(3)}x_2^{(3)} + q^{-1}(q-1)x_4x_6x_2^{(3)} \\
& +q^{\frac{1}{2}}(q-1)x_3^2x_4x_5x_6x_1^{(3)}x_2^{(3)} + q^{\frac{3}{2}}(q-1)x_3^2x_4x_5x_6x_1^{(3)}x_2^{(3)}.
\end{aligned}$$

Some Properties of Generalized Cluster Algebras of Geometric Type

Theorem
Let $\Sigma = (\widetilde{\mathbf{x}}, \rho, \widetilde{B})$ *be an acyclic and coprime generalized seed. Then we have*

$$\mathcal{A}(\Sigma) = \mathcal{L}^{(n)}(\Sigma).$$

Definition
A monomial in $x_1, x_1^{(n)}, \ldots, x_n, x_n^{(n)}$ is called a projective standard monomial if it contains no product of the form $x_i x_i^{(n)}$ for any $i \in [1, n]$.

Theorem
Let $\Sigma = (\widetilde{\mathbf{x}}, \rho, \widetilde{B})$ *be an acyclic and coprime generalized seed. Then the projective standard monomials in* $x_1, x_1^{(n)}, \ldots, x_n, x_n^{(n)}$ *form a* $\mathbb{Z}\mathbb{P}$*-basis of* $\mathcal{A}(\Sigma)$*.*

A Criterion of Generalized Cluster Algebras without 3-cycles

Theorem
If the standard monomials in $x_1, x_1', \ldots, x_n, x_n'$ are linearly independent over $\mathbb{ZP}$, then the directed graph $\Gamma\left(\widetilde{\mathbf{x}}, \rho, \widetilde{B}\right)$ does not contain 3-cycles.

Remark
In classical cluster algebras, Berenstein, Fomin and Zelevinsky proved that standard monomials in $\{x_1, x_1', \ldots, x_n, x_n'\}$ are linearly independent over $\mathbb{ZP}$ if and only if the directed graph associated to the seed $\left(\widetilde{\mathbf{x}}, \widetilde{B}\right)$ is acyclic. The sufficient part was extended to the case of generalized cluster algebras of geometric type by Bai, Chen, Ding and Xu.

Thanks for listening!