Model structure from one cotorsion pair

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- The Hovey correspondence constructs model structures via two cotorsion pairs
- This is to report a way using only one cotorsion pair, based on Beligiannis and Reiten [BR]

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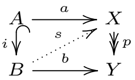
(Closed) Model structures

Def. (D. Quillen) A model structure on category $\mathcal{M}$ is a triple (Cofib($\mathcal{M}$), Fib($\mathcal{M}$), Weq($\mathcal{M}$)) of classes of morphisms (called cofibrations, fibrations, weak equivalences, resp.), satisfying:

(**Two out of three axiom**) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$. If two of f, g, gf are weak equivalence, so is the third.

(**Retract axiom**) If f is a retract of g and g is a cofibration (fib., weak equivalence), so is f.

(**Lifting axiom**) For a commutative square with $i \in \mathrm{Cofib}(\mathcal{C})$, $p \in \mathrm{Fib}(\mathcal{C})$



If $i \in \mathrm{Weq}(\mathcal{C})$ or $p \in \mathrm{Weq}(\mathcal{C})$, then $\exists\ s : B \longrightarrow X$ s.t. the two triangles commute.

(**Factorization axiom**) Any morphism f has factorizations $f = pi = qj$, where $i \in \mathrm{Cofib}(\mathcal{C}) \cap \mathrm{Weq}(\mathcal{C})$, $p \in \mathrm{Fib}(\mathcal{C})$, $j \in \mathrm{Cofib}(\mathcal{C})$, $q \in \mathrm{Fib}(\mathcal{C}) \cap \mathrm{Weq}(\mathcal{C})$.

5 classes of morphisms + 5 classes of objects

Notations. $(\mathrm{Cofib}(\mathcal{M}), \mathrm{Fib}(\mathcal{M}), \mathrm{Weq}(\mathcal{M}))$: a model structure on category $\mathcal{M}$ with 0 objects

(1) *The class of trivial cofibrations*: $\mathrm{Cofib}(\mathcal{M}) \cap \mathrm{Weq}(\mathcal{M})$.

(2) *The class of trivial fibrations*: $\mathrm{Fib}(\mathcal{M}) \cap \mathrm{Weq}(\mathcal{M})$.

(3) *The class of cofibrant objects*: $\mathcal{C} = \{X \in \mathcal{M} \mid 0 \to X \text{ is a cofibration}\}$.

(4) *The class of fibrant objects.*: $\mathcal{F} = \{X \in \mathcal{M} \mid Y \to 0 \text{ is a fibration}\}$.

(5) *The class of trivial objects*: $\mathcal{W} = \{X \in \mathcal{M} \mid 0 \to W \text{ is a weak equivalence}\} = \{X \in \mathcal{M} \mid W \to 0 \text{ is a weak equivalence}\}$.

(6) *The class of trivial cofibrant objects*: $\mathcal{C} \cap \mathcal{W}$.

(7) *The class of trivial fibrant objects*: $\mathcal{F} \cap \mathcal{W}$.

Quillen's homotopy category $\mathrm{Ho}(\mathcal{M})$

Def. $(\mathrm{Cofib}(\mathcal{M}),\ \mathrm{Fib}(\mathcal{M}),\ \mathrm{Weq}(\mathcal{M}))$: a model structure on category $\mathcal{M}$. The homotopy category $\mathrm{Ho}(\mathcal{M})$ is the localization $\mathrm{Weq}(\mathcal{M})^{-1}\mathcal{M}$.

Theorem (D. Quillen; A. Beligiannis, I. Reiten)

If $(\mathrm{Cofib}(\mathcal{M}),\ \mathrm{Fib}(\mathcal{M}),\ \mathrm{Weq}(\mathcal{M}))$ is a model structure on additive category $\mathcal{M}$, then $\mathrm{Ho}(\mathcal{M})$ is pretriangulated.

Exact categories

Def. (Quillen; Keller) An exact category is a pair $(\mathcal{A}, \mathcal{E})$, where $\mathcal{A}$ is an additive category, and $\mathcal{E}$ is a class of exact pairs satisfying the axioms below, where an exact pair $(i, d) \in \mathcal{E}$ is called a **conflation**, i an **inflation**, and d a **deflation**.

(E0) $\mathcal{E}$ is closed under isomorphisms, and Id_0 is a deflation.

(E1) The composition of two deflations is a deflation.

(E2) For a deflation $d : Y \to Z$ and a morphism $f : Z' \to Z$, there is a pullback

$$\begin{array}{ccc} Y' & \overset{d'}{\dashrightarrow} & Z' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{d} & Z \end{array}$$

such that d' is a deflation.

(E2$^{\mathrm{op}}$) For an inflation $i : X \to Y$ and a morphism $f : X \to X'$, there is a pushout

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X' & \overset{i'}{\dashrightarrow} & Y' \end{array}$$

such that i' is an inflation.

A sequence $0 \to X \overset{i}{\to} Y \overset{d}{\to} Z \to 0$ is **admissible exact** if (i, d) is a conflation.

Weakly idempotent complete exact categories

Prop. ([Keller etc.]; [Bühler, 7.2, 7.6]) Let $\mathcal{A}$ be an exact category. Then the following are equivalent:

(i) Any splitting epimorphism in $\mathcal{A}$ is a deflation.

(ii) Any splitting epimorphism in $\mathcal{A}$ has a kernel.

(iii) Any splitting monomorphism in $\mathcal{A}$ is an inflation.

(iv) Any splitting monomorphism in $\mathcal{A}$ has a cokernel.

(v) If de is a deflation, then so is d.

(vi) If ki is an inflation, then so is i.

An exact category satisfying the above equivalent conditions in Prop. is called **a weakly idempotent complete** exact category (T. Thomason, T. Trobaugh).

Any full subcategory of an abelian category which is closed under extensions and direct summands is a weakly idempotent complete exact category, but **not** abelian in general.

For example, $\mathcal{GP}(\mathcal{A})$.

Cotorsion pairs in exact categories

Def. $\mathcal{A}$: an exact category.

A pair $(\mathcal{C}, \mathcal{F})$ of classes of objects is a cotorsion pair, if

$$\mathcal{C} = {}^{\perp}\mathcal{F} = \{X \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(X, \mathcal{F}) = 0\}, \qquad \mathcal{F} = \mathcal{C}^{\perp}.$$

(2) A cotorsion pair $(\mathcal{C}, \mathcal{F})$ is **hereditary**, if

- $\mathcal{C}$ is closed under the kernel of deflations; and
- $\mathcal{F}$ is closed under cokernel of inflations.

(3) A cotorsion pair $(\mathcal{C}, \mathcal{F})$ is **complete** if for any object $X \in \mathcal{A}$ there are exact sequences

$$0 \to F \to C \to X \to 0, \quad 0 \to X \to F' \to C' \to 0, \text{ where } C,\ C' \in \mathcal{C},\ F,\ F' \in \mathcal{F}.$$

Prop. ([Šťovíček, 6.17]) Let $(\mathcal{C}, \mathcal{F})$ be a complete cotorsion pair in a weakly idempotent complete exact category $\mathcal{A}$. Then the following are equivalent:

(1) $(\mathcal{C}, \mathcal{F})$ is hereditary;

(2) $\mathcal{C}$ is closed under the kernel of deflations;

(3) $\mathcal{F}$ is closed under the cokernel of inflations;

(4) $\mathrm{Ext}^2_{\mathcal{A}}(C, F) = 0$ for all $C \in \mathcal{C}$ and $F \in \mathcal{F}$;

(5) $\mathrm{Ext}^i_{\mathcal{A}}(C, F) = 0$ for all $C \in \mathcal{C}$, $F \in \mathcal{F}$, and $i \geq 2$.

Exact model structures

Def. (M. Hovey; J. Gillespie) A model structure $(\mathrm{Cofib}(\mathcal{A}),\ \mathrm{Fib}(\mathcal{A}),\ \mathrm{Weq}(\mathcal{A}))$ on exact category $\mathcal{A}$ is **exact**, if

(i) $\mathrm{Cofib}(\mathcal{A}) = \{\text{inflation } f \mid \mathrm{Cok}\, f \text{ is a cofibrant object}\}$

(ii) $\mathrm{Fib}(\mathcal{A}) = \{\text{deflation } f \mid \mathrm{Ker}\, f \text{ is a fibrant object}\}$.

In this case one has

$\mathrm{Cofib}(\mathcal{A}) \cap \mathrm{Weq}(\mathcal{A}) = \{\text{inflation } f \mid \mathrm{Cok}\, f \text{ is a trivial cofibrant object}\}$;

$\mathrm{Fib}(\mathcal{A}) \cap \mathrm{Weq}(\mathcal{A}) = \{\text{deflation } f \mid \mathrm{Ker}\, f \text{ is a trivial fibrant object}\}$.

Hovey correspondence

Def. A triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of classes of objects of exact category $\mathcal{A}$ is **a Hovey triple**, if

(i) $\mathcal{Z}$ is **thick**, i.e. $\mathcal{Z}$ is closed under direct summands, and if two terms in an admissible exact sequence $0 \to X \to Y \to Z \to 0$ are in $\mathcal{Z}$, then so is the third one.

(ii) $(\mathcal{X},\ \mathcal{Y} \cap \mathcal{Z})$ and $(\mathcal{X} \cap \mathcal{Z},\ \mathcal{Y})$ are complete cotorsion pairs.

Theorem (M. Hovey, 2002; Gillespie, 2011; Šťovíček, 2014) There is an one-one correspondence between exact model structure and the Hovey triples in weakly idempotent complete exact cat. $\mathcal{A}$:

$$(\mathrm{Cofib}(\mathcal{A}),\ \mathrm{Fib}(\mathcal{A}),\ \mathrm{Weq}(\mathcal{A})) \mapsto$$

(the cofibrant objects. class, the fibrant objects. class, the trivial objs. class)

the inverse: Hovey triple $(\mathcal{C},\ \mathcal{F}, \mathcal{W}) \mapsto (\mathrm{Cofib}(\mathcal{A}),\ \mathrm{Fib}(\mathcal{A}),\ \mathrm{Weq}(\mathcal{A}))$, where

$$\mathrm{Cofib}(\mathcal{A}) = \{\text{inflation with cokernel in } \mathcal{C}\}$$

$$\mathrm{Fib}(\mathcal{A}) = \{\text{deflation with kernel in } \mathcal{F}\},$$

$$\mathrm{Weq}(\mathcal{A}) = \{pi \ \mid\ i \text{ inflation},\ \mathrm{Cok}\, i \in \mathcal{C} \cap \mathcal{W},\ p \text{ deflation},\ \mathrm{Ker}\, p \in \mathcal{F} \cap \mathcal{W}\}.$$

5 morphism classes in a weakly idempotent complete exact cat.

$\mathcal{A}$: a weakly idempotent complete exact category

$\mathcal{X}$, $\mathcal{Y}$: additive full subcategory closed under direct summands and isomorphisms

$\omega := \mathcal{X} \cap \mathcal{Y}$

CoFib_ω : the class of inflations f with $\mathrm{Cok}\, f \in \mathcal{X}$.

Fib_ω: the class of morphisms which is ω-epic.

Weq_ω: the class of morphisms $f : A \to B$ s.t. there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle \binom{1}{0}} \searrow \quad \nearrow {\scriptstyle (f,t)} & \\ & A \oplus W & \end{array}$$

with $W \in \omega$, (f, t) a deflation, $\mathrm{Ker}(f, t) \in \mathcal{Y}$.

TCoFib_ω : the class of splitting monomorphism f with $\mathrm{Cok}\, f \in \omega$.

TFib_ω : the class of deflations f with $\mathrm{Ker}\, f \in \mathcal{Y}$.

Thus $\mathrm{Weq}_\omega = \{gf \mid f \in \mathrm{TCoFib}_\omega,\ g \in \mathrm{TFib}_\omega\}$.

Important Fact: $\mathrm{TCoFib}_\omega = \mathrm{CoFib}_\omega \cap \mathrm{Weq}_\omega$, $\quad \mathrm{TFib}_\omega = \mathrm{Fib}_\omega \cap \mathrm{Weq}_\omega$.

Main result

Theorem. Let $\mathcal{A}$ be a weakly idempotent complete exact category

$\mathcal{X}$, $\mathcal{Y}$ add. full subcat. of $\mathcal{A}$ closed under direct summands and isomorphisms, $\omega := \mathcal{X} \cap \mathcal{Y}$.

Then $(\mathrm{CoFib}_\omega,\ \mathrm{Fib}_\omega,\ \mathrm{Weq}_\omega)$ is a model structure iff $(\mathcal{X}, \mathcal{Y})$ is a **hereditary** complete cotorsion pair, and ω is contravariantly finite in $\mathcal{A}$.

If this is the case, then

the class $\mathcal{C}_\omega$ of cofibrant objects is $\mathcal{X}$,

the class $\mathcal{F}_\omega$ of fibrant objects is $\mathcal{A}$,

the class $\mathcal{W}_\omega$ of trivial objects is $\mathcal{Y}$,

the homotopy category $\mathrm{Ho}(\mathcal{A})$ is $\mathcal{X}/\omega$.

Remark: This is due to Beligiannis and Reiten (Mem. AMS, 2007), in abelian categories.; but condition "hereditary" is missing.

The heredity of $(\mathcal{X}, \mathcal{Y})$ is necessary

Lemma. $\mathcal{A}$: a weakly idempotent complete exact cat.

$(\mathcal{X}, \mathcal{Y})$ a complete cotorsion pair with $\omega = \mathcal{X} \cap \mathcal{Y}$ contravariantly finite.

If $(\mathrm{CoFib}_\omega, \mathrm{Fib}_\omega, \mathrm{Weq}_\omega)$ is a model structure, then $(\mathcal{X}, \mathcal{Y})$ is hereditary.

Proof. To show: $\mathcal{Y}$ is closed under the cokernel of inflation. For an admissible exact seq. $0 \to Y_1 \to Y_2 \xrightarrow{p} M \to 0$ with $Y_1, Y_2 \in \mathcal{Y}$, by definition $0 : Y_2 \to 0$ is in TFib_w, and $p \in \mathrm{TFib}_w$. Thus $(Y_2 \to 0) \in \mathrm{Weq}_\omega$, $\quad p \in \mathrm{Weq}_\omega$. By

$$(Y_2 \to 0) = (M \to 0) \circ p$$

one sees that $M \to 0$ is in Weq_ω. Thus one has the decomposition

$$\begin{array}{ccc} M & \xrightarrow{\quad 0 \quad} & 0 \\ & {\scriptstyle \binom{1}{0}} \searrow \quad M \oplus W \quad \nearrow {\scriptstyle 0} & \end{array}$$

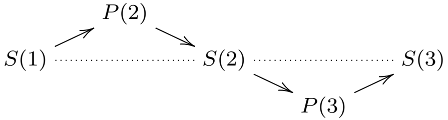
with $W \in \omega$ and $M \oplus W \to 0$ in TFib_w. Thus $M \oplus W \in \mathcal{Y}$, and hence $M \in \mathcal{Y}$. $\square$

The heredity of $(\mathcal{X}, \mathcal{Y})$ is necessary (continued)

A **non hereditary** complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$ with $\omega := \mathcal{X} \cap \mathcal{Y}$ contravariantly finite:

Example k : a field, $\quad Q$: the quiver $3 \overset{\beta}{\to} 2 \overset{\alpha}{\to} 1$, $\quad A = kQ/\langle \alpha\beta \rangle$

$\mathcal{C} := \mathrm{add}({}_AA \oplus S(3))$. $\quad$ The Auslander-Reiten quiver of A :



- $(\mathcal{C}, \mathcal{C})$ is a complete cotorsion pair in the left A-module category A-mod
- $\omega := \mathcal{C} \cap \mathcal{C} = \mathcal{C}$ is contravariantly finite in A-mod
- The cotorsion pair $(\mathcal{C}, \mathcal{C})$ is **Not** hereditary, since there is an exact sequence

$$0 \longrightarrow S(2) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0$$

or, since $\mathrm{Ext}^2_A(S(3), S(1)) \neq 0$. So $(\mathrm{CoFib}_\omega,\ \mathrm{Fib}_\omega,\ \mathrm{Weq}_\omega)$ is **Not** a model structure

When the model structure is exact?

When the model structure $(\mathrm{CoFib}_\omega, \mathrm{Fib}_\omega, \mathrm{Weq}_\omega)$ can be got by the Hovey correspondence? or equivalently, when it is exact?

Prop. $\mathcal{A}$: a weakly idempotent complete exact category $(\mathcal{X}, \mathcal{Y})$ a hereditary complete cotorsion pair with $\omega = \mathcal{X} \cap \mathcal{Y}$ contravariantly finite.

Then the model structure $(\mathrm{CoFib}_\omega, \mathrm{Fib}_\omega, \mathrm{Weq}_\omega)$ is exact $\Longleftrightarrow$ $\mathcal{A}$ has enough projective objects and $\omega = \mathcal{P}$.

Proof. $\Longleftarrow$: If $\omega = \mathcal{P}$ and $f : A \to B$ is ω-epic, taking a deflation $g : P \to B$, then $g = fh$ with $h : P \to A$. Since $(\mathcal{A}, \mathcal{E})$ is weakly idempotent complete, f is a deflation. So $\{f \mid f \text{ is } \omega\text{-epic}\} = \{\text{deflation}\}$.

$\Longrightarrow$: Assume $\{f \mid f \text{ is } \omega\text{-epic}\} = \{\text{deflation}\}$. Then $(\mathrm{CoFib}_\omega, \mathrm{Fib}_\omega, \mathrm{Weq}_\omega)$ is an abelian model structure. By Hovey correspondence

(the cofibration objects class, the fibration objects class, the trivial objects class)

is a Hovey triple, i.e.,

$(\mathcal{X}, \mathcal{A}, \mathcal{Y})$ is a Hovey triple, thus $(\omega, \mathcal{A})$ is a cotorsion pair, so $\omega = \mathcal{P}$. □

The model structure via one cotorsion pair is **not** exact in general

A : Artin algebra $\quad T$: a tilting module

Then T is a tilting object in exact category $\mathcal{P}^{<\infty}$, i.e., $\mathrm{Ext}^i_\Lambda(T,T)=0$ for $i\geq 1$, and $\mathrm{Thick}(T)=\mathcal{P}^{<\infty}$.

$\mathcal{P}^{<\infty}$ is weakly idempotent complete; and it is **not** abelian iff gl.dim$\Lambda=\infty$.

$\widetilde{\mathrm{add}(T)}=\{M \mid \exists$ exact seq. $0\to M\to T_0\to\cdots\to T_s\to 0,\ T_i\in\mathrm{add}(T)\}$
$\widehat{\mathrm{add}(T)}=\{M \mid \exists$ exact seq. $0\to T_s\to\cdots\to T_0\to M\to 0,\ T_i\in\mathrm{add}(T)\}$

Then $(\widetilde{\mathrm{add}T},\ \widehat{\mathrm{add}T})$ is a hereditary complete cotorsion pair in $\mathcal{P}^{<\infty}$, with $\omega := \widetilde{\mathrm{add}T}\cap\widehat{\mathrm{add}T}=\mathrm{add}T$ contravariantly finite in $\mathcal{P}^{<\infty}$.([H. Krouse 7.2])

If T is not projective, then the model structure on $\mathcal{P}^{<\infty}$ induced by $(\widetilde{\mathrm{add}T},\ \widehat{\mathrm{add}T})$ is **not** exact.

Weakly projective model structures

Proposition Let (CoFib, Fib, Weq) be a model structure on exact category $\mathcal{A}$. Then the following are equivalent.

(1) Cofibrations are exactly inflations with cofibrant cokernel, and any trivial fibration is a deflation.

(2) $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{C}, \mathrm{T}\mathcal{F}) = 0$, any cofibration is an inflation, and any trivial fibration is a deflation.

(3) Trivial fibrations are exactly deflations with trivially fibrant kernel, and any cofibration is an inflation.

(4) Cofibrations are exactly inflations with cofibrant cokernel, and trivial fibrations are exactly deflations with trivially fibrant kernel.

Moreover, if in addition $\mathcal{A}$ is weakly idempotent complete, then all the conditions above are equivalent to

(5) $(\mathcal{C}, \mathrm{T}\mathcal{F})$ is a complete cotorsion pair.

Definition A model structure on an exact category is **weakly projective,** if any object is fibrant and it satisfies the conditions in the Proposition above.

The correspondence of Beligiannis and Reiten

Theorem $\mathcal{A}$: a weakly idempotent complete exact category

S_C: the class of hereditary complete cotorsion pairs $(\mathcal{X}, \mathcal{Y})$ with $\omega = \mathcal{X} \cap \mathcal{Y}$ contravariantly finite

S_M: the class of weakly projective model structure on $\mathcal{A}$

Then

$$\Phi : (\mathcal{X}, \mathcal{Y}) \mapsto (\mathrm{CoFib}_\omega, \mathrm{Fib}_\omega, \mathrm{Weq}_\omega)$$

and

$$\Psi : (\mathrm{CoFib}, \mathrm{Fib}, \mathrm{Weq}) \mapsto (\mathcal{C}, \mathrm{T}\mathcal{F})$$

give a bijection between S_C and S_M.

Thus, the intersection of Hovey's exact model structures and Beligiannis and Reiten's weakly projective model structures, on a weakly idempotent complete exact category, is exactly the **projective** model structures.

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Thank you very much!