Categorification and mirror symmetry for Gr(k, n)

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Plan of this talk:

- The Grassmannian cluster category CM C
- The invariant $\kappa(M, N)$
- Cluster characters, partition functions and flow polynomials.
- Rietsch-Williams' mirror symmetry for the Grassmannian $\mathrm{Gr}(k, n)$.
- Cones of $\underline{\kappa}$, g-vectors and the potential W (categorical interpretation of RW's mirror symmetry).

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The Grassmannian cluster category CM C

- $R = \mathbb{C}[[t]]$ the formal power series ring in t.
- $C = RQ/\mathcal{I}$
  - $Q =$ the double cyclic quiver with clockwise arrows x and anti-clockwise arrows y. The *trivial path* at vertex i is denoted by e_i.
  - the relations $\mathcal{I}$: $xy = yx = t$ and $x^k = y^{n-k}$.
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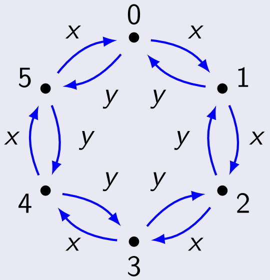
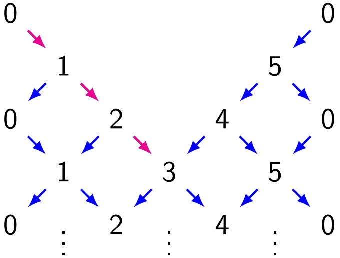
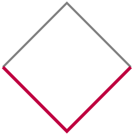
Example: $k = 3$ and $n = 6$, $xy = yx = t$, $x^3 = y^3$



Rank one modules M in CM C, i.e. $e_i M \cong R$ for all i



$P_0 = M_{123}$



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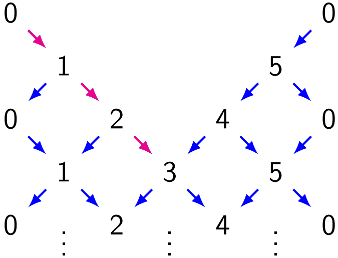

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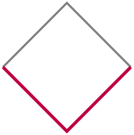
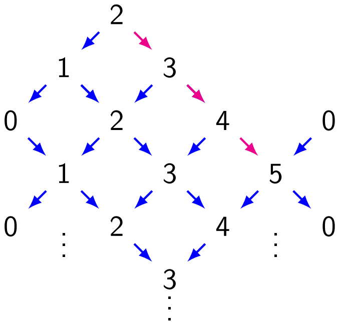
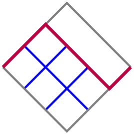
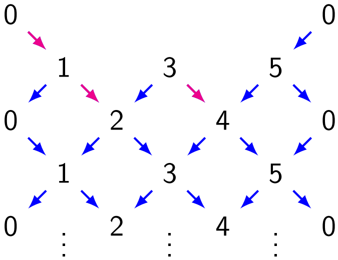


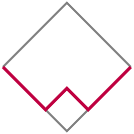


$P_2 = M_{345}$





M_{124}



Remark:

- Rank one modules are parameterised by k-subsets of $[n] = \{1, \ldots, n\}$. Denote the module corresponding to I by M_I.
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- There is a well-defined rank for any $M \in \mathrm{CM}\, C$, given by

$$\mathrm{rank}\, M = \mathrm{rank}_R\, e_i M.$$

- CM C is of finite type, if and only if $(k, n) = (2, n), (3, 6), (3, 7)$ or $(3, 8)$. So in other types, there are infinitely many indecomposable modules of rank > 1.

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- CM C is a Frobenius 2-CY category.
- CM C provides an additive categorification of the cluster structure on the Grassmannian $\mathrm{Gr}(k, n)$.

The restriction functor and its adjoints

- $\mathrm{e}_0 : \mathrm{CM}\, C \to \mathrm{CM}\, R$ (free R-modules) the restriction functor, $M \mapsto e_0 M$ (note: e_0 the trivial path at vertex 0).
- $\mathrm{P} : \mathrm{CM}\, R \to \mathrm{CM}\, C,\ \ W \mapsto Ce_0 \otimes_R W$.
- $\mathrm{J} : \mathrm{CM}\, R \to \mathrm{CM}\, C,\ \ W \mapsto \mathrm{Hom}_R(e_0 C, W)$.
- P and J are left and right adjoints to e_0.
- $\alpha\colon\ \mathrm{P}\,\mathrm{e}_0\, M \to M$ and $\beta\colon N \to \mathrm{J}\,\mathrm{e}_0\, N$ are embeddings with finite dimensional quotients, since $\mathrm{e}_0\,\mathrm{P}\,\mathrm{e}_0\, M = \mathrm{e}_0\, M$ and $\mathrm{e}_0\,\mathrm{J}\,\mathrm{e}_0\, N = \mathrm{e}_0\, N$.

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Let $\phi_0\colon\; \mathrm{Hom}_C(M, N) \to \mathrm{Hom}_R(e_0 M, e_0 N)$, $f = (f_i) \mapsto f_0$, and let

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Proposition [JKS] Let $M, N \in \mathrm{CM}\, C$.

- $\mathrm{cok}\, \phi_0 \cong \mathrm{cok}\, \mathrm{Hom}(\alpha, N) \cong \mathrm{cok}\, \mathrm{Hom}(M, \beta) \leq \mathrm{Hom}(M, \mathrm{J}\, e_0\, N/N)$.
- $K(M, N)$ is an $(\mathrm{End}\, M)^{\mathrm{op}}$-module.
- $\dim K(M, N) < \infty$.

The invariant κ

Definition

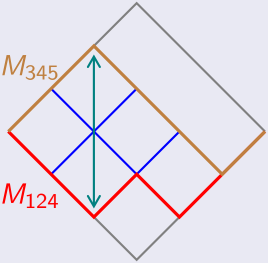
- Let $\kappa(M, N) = \dim K(M, N)$.
- When $M = \oplus_i M_i$, let $\underline{\kappa}(M, N) = (\kappa(M_i, N))_i$.

Remark: When each M_i is indecomposable, then $\underline{\kappa}(M, N)$ is the dimension vector of $K(M, N)$ as an $(\operatorname{End} M)^{\mathrm{op}}$-module.

Remark: $\kappa(P_0, N) = 0$ for all $N \in \mathrm{CM}\, C$, since $\mathrm{P}\, e_0\, P_0 = P_0$ and so $\operatorname{Hom}(P_0, N) = \operatorname{Hom}(\mathrm{P}\, e_0\, P_0, N)$.

Example

$$\kappa(M_{124}, M_{345}) = 0,$$
$$\kappa(M_{345}, M_{124}) = 2.$$



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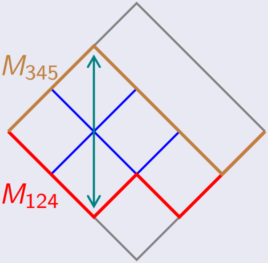

Remark: Denote the Young diagram corresponding to I by λ_I. In general, we have $\kappa(M_I, M_J) = \mathrm{MaxDiag}(\lambda_J \backslash \lambda_I)$, i.e. the maximal length of the diagonals in $\lambda_J \backslash \lambda_I$.

Weakly separated sets and the rectangle cluster tilting object

- Two sets $I, J \subseteq [n]$ of size k are said to be *weakly separated* (or *non-crossing*) if there are no $a, c \in I \backslash J$ and $b, d \in J \backslash I$ such that a, b, c, d are cyclically ordered.
- $\mathrm{Ext}^1(M_I, M_J) = 0$ if and only if I, J are weakly separated.
- Let $\mathcal{S}$ be a collection of weakly separated sets. $T_{\mathcal{S}} = \oplus_{I \in \mathcal{S}} M_I$ is a cluster tilting object (CTO) if and only if $\mathcal{S}$ is maximal.

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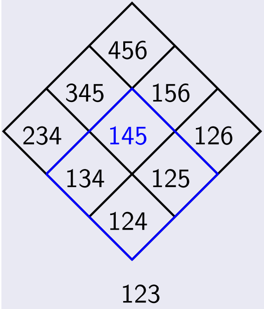
- Let $\mathcal{S}_\square$ be the collection of the labels of the boxes (including the empty box) in the $k \times (n-k)$-grid. Then $\mathcal{S}_\square$ is a maximal collection of weakly separated sets.
- Let $T_\square = \oplus_{I \in \mathcal{S}_\square} M_I$. Then $T_\square$ is a CTO and is called a *rectangle cluster tilting object*.

The rectangle cluster for Gr(3, 6) and $\underline{\kappa}(T_\square, M_{134})$

$T_\square$:



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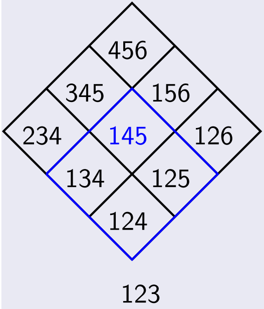

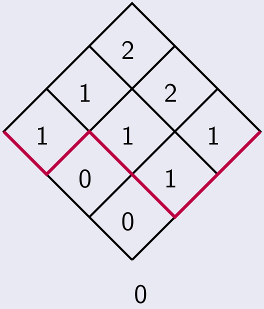

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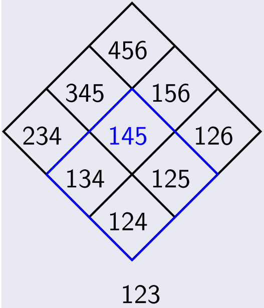
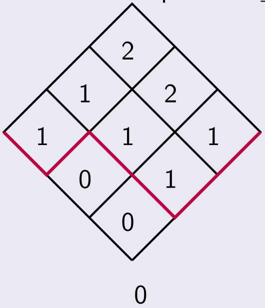


A Gelfand-Tsetlin-2-pattern: $\underline{\kappa}(T_\square, M_{134})$



Proposition [JKS]

$\{\underline{\kappa}(T_\square, M) \mid \mathrm{rank}\, M \leq r\} = \{\text{GT-}r\text{-patterns}\}.$

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Remark: When I, J are weakly separated k-sets, $\kappa(M_I, M_J) - \kappa(M_J, M_I)$ computes the quasi-commutation rule for the quantum minors Δ_I^q and Δ_J^q. So CM C provides a categorical model for the quantum Grassmannian $\mathbb{C}_q[\mathrm{Gr}(k, n)]$.

Mutations of T and $\underline{\kappa}(T, M)$

Let $T = \oplus_i T_i$, each T_i is indecomposable. There are two mutation sequences associated to each mutable summand T_j:

$$0 \longrightarrow T_j^* \longrightarrow E_j \longrightarrow T_j \longrightarrow 0$$

and

$$0 \longrightarrow T_j \longrightarrow F_j \longrightarrow T_j^* \longrightarrow 0$$

Let $T' = \oplus_{i \neq j} T_i \oplus T_j^*$. Then T' is again a CTO.

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Theorem [JKS]

Let $M \in \mathrm{CM}\, C$ be generic (e.g. when $\mathrm{Ext}^1(M, M) = 0$). Then

$$\kappa(T_j^*, M) + \kappa(T_j, M) = \min\{\kappa(E_j, M), \kappa(F_j, M)\}.$$

Consequently, the map $\underline{\kappa}(T, M) \mapsto \underline{\kappa}(T', M)$ is a tropical $\mathbb{A}$-mutation.

Cluster characters

Let $\Psi : \mathrm{CM}\, C \to \mathbb{C}[\mathrm{Gr}(k,n)]$ be the cluster character such that $\Psi_{M_I} = \Delta_I$, the minor indexed by I. We have

$$\Psi_M \Psi_N = \Psi_{M\oplus N} \text{ and } \Psi_{T_j}\Psi_{T_j^*} = \Psi_{E_j} + \Psi_{F_j},$$

where $M, N, T_j, T_j^* \in \mathrm{CM}\, C$, E_j, F_j are the middle terms in the mutation sequences for T_j, T_j^*.

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Generalised partition functions

Define

$$\mathcal{P}_M = (\mathcal{P}_M^T) = \sum_{X \in D_M} \chi(\mathrm{Gr}_{d_{[X]}}(\underline{\mathrm{Hom}}(T, M)))x^{[X]},$$

where $\mathrm{Gr}_{d_{[X]}}(\underline{\mathrm{Hom}}(T, M))$ is the Grassmannian of $d_{[X]}$ dim. submodules of $\underline{\mathrm{Hom}}(T, M)$ and $d_{[X]}$ depends on the class $[X]$ of X. Roughly, D_M consists of classes of submodules of $\mathrm{Hom}(T, M)$ that are M when restricted to C.

Theorem [JKS]

- When $T = T_{\mathcal{S}}$ and $M = M_I$, $\mathcal{P}_M$ is the partition function associated to I, which is defined combinatorially on a (plabic) graph corresponding to T.
- $\mathcal{P}_M = \frac{\tilde{\Psi}_{\Omega M}}{\tilde{\Psi}_{P_M}}$, where $0 \to \Omega_M \to P_M \to M \to 0$ is any projective presentation of M.
- $\mathcal{P}_M$ is a cluster character. Consequently, the (classical) partition functions $\mathcal{P}_{M_I}$ satisfy Plücker relations.

Weight modules

Let $A = (\operatorname{End} T)^{\mathrm{op}}$. Note that C is a summand of T. Let $\mathrm{e} : \mathrm{CM}\, A \to \mathrm{CM}\, C$ be the restriction functor, $X \mapsto \mathrm{e}\, X$.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & \operatorname{Hom}(T, \mathrm{e}\, X) & \longrightarrow & \mathrm{G}X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & \operatorname{Hom}(T, \mathrm{J}\, \mathrm{e}_0\, X) & \longrightarrow & \mathrm{Wt}\, X & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{K}(T, \mathrm{e}\, X) & = & \mathrm{K}(T, \mathrm{e}\, X) & & \\
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- $\mathrm{Wt}\, X$ is called a weight module of X.
- $\mathrm{Wt} : \mathrm{CM}\, A \to \mathrm{fd}\, A$ is exact. So it induces a map $\mathrm{wt} : K(\mathrm{CM}\, A) \to K(\mathrm{fd}\, A)$

The generalised flow polynomial

Define

$$\mathcal{F}_M = \sum_{X \in D_M} \chi(\mathrm{Gr}^d(\underline{\mathrm{Hom}}(T, M)))x^{[\mathrm{Wt}\, X]}.$$

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Theorem [JKS]

- $\mathcal{F}_M$ is a cluster character.
- $\mathcal{F}_M$ has a unique minimal term $x^{\underline{\kappa}(T,M)}$.
- When $T = T_{\mathcal{S}}$, $\mathcal{F}_{M_I}$ is the flow polynomial associated to I defined on the plabic graph corresponding to T.
- $\underline{\kappa}(T, M_I)$ is RW's valuation of the minor Δ_I, i.e. $\mathrm{val}_G(\Delta_I)$. Moreover,

$$\{\underline{\kappa}(T, M) \colon M \in \mathrm{CM}\ C\} = \{\mathrm{val}_G(f) : f \in \mathbb{C}[\mathrm{Gr}(k, n)] \text{ is pointed}\}.$$

Rietsch-Williams' mirror symmetry for Grassmannians

Newton-Okounkov body $\Delta(G)$ and the potential polytope $\Delta(W)$

$\mathbb{X}$ side:

- $X = \mathrm{Gr}(k, n)$
- Newton-Okounkov body $\Delta(G)$ via the valuation val_G e.g. $\mathrm{val}_G(\Delta_I)$=exponent of minimal term of the flow polynomial ass. to I
- (G= a seed obtained from a plabic graph by mutation.)

$\mathbb{A}$ side:

- $\check{X}$ (mirror dual of X) with the potential W constructed by Marsh-Rietsch.
- Express the potential W in the cluster associated to G to obtain a Laurent polynomial W_G
- Tropicalise W_G to obtain a polytope $\Delta(W_G)$

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Theorem [RW]

$$\Delta(G) = \Delta(W_G) \text{ (or equivalently } \mathrm{Cone}(G) = \mathrm{Cone}(W_G)).$$

Marsh-Rietsch's superpotential W

$$W = q\frac{\Delta_{\hat{J}_{n-k}}}{\Delta_{J_{n-k}}} + \sum_{i\neq n-k} \frac{\Delta_{\hat{J}_i}}{\Delta_{J_i}},$$

where $J_i = [i+1, i+k]$ and $\Delta_{\hat{J}_i} = [i+1, i+k-1] \cup \{i+k+1\}$. For instance, when $k = 2$ and $n = 5$,

$$W = \frac{\Delta_{13}}{\Delta_{12}} + \frac{\Delta_{24}}{\Delta_{23}} + \frac{\Delta_{35}}{\Delta_{34}} + q\frac{\Delta_{14}}{\Delta_{45}} + \frac{\Delta_{25}}{\Delta_{15}}$$

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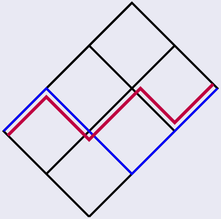
where $J_i = [i+1, i+k]$ and $\Delta_{\hat{J}_i} = [i+1, i+k-1] \cup \{i+k+1\}$.
For instance, when $k = 2$ and $n = 5$,

$$W = \frac{\Delta_{13}}{\Delta_{12}} + \frac{\Delta_{24}}{\Delta_{23}} + \frac{\Delta_{35}}{\Delta_{34}} + q\frac{\Delta_{14}}{\Delta_{45}} + \frac{\Delta_{25}}{\Delta_{15}}$$

Example

$J = [2,3] = \{2,3\}$ $(M_J = P_1)$

$\hat{J} = [2,2] \cup \{4\} = \{2,4\}$

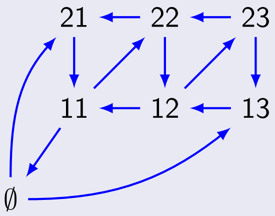


Example: $T_\square$ and its Gabriel quiver for $k = 5$ and $n = 2$

23	34	45
13	14	15

12

Rename the summands, e.g. T_{ij} is the summand at row i and col j



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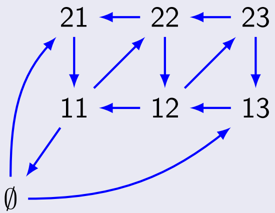

Using Fu-Keller's cluster character formula, we express the numerators in W as Laurent polynomials in the initial cluster variables $\Delta_{ij} = \Psi_{T_{ij}}$, denoted by p_{ij}.

$$W = \frac{p_{11}}{p_{\emptyset}} + \frac{p_{21}}{p_{11}} + \frac{p_{13}}{p_{12}} + \frac{p_{12}}{p_{11}} + \frac{p_{22}p_{\emptyset}}{p_{11}p_{12}} + \frac{p_{23}p_{11}}{p_{12}p_{22}} + \frac{p_{22}p_{\emptyset}}{p_{11}p_{21}} + \frac{p_{23}p_{\emptyset}}{p_{12}p_{13}} + q\frac{p_{12}}{p_{23}}.$$

This provides an alternative explanation to MR's original proof.

Tropicalising W gives the following inequalities:

$$0 \leq x_{\emptyset} \leq x_{11} \leq x_{12} \leq x_{13}, \; x_{11} \leq x_{21}, \; x_{12} - x_{\emptyset} \leq x_{22} - x_{11} \leq x_{23} - x_{12} \leq r,$$

$$x_{21} - x_{\emptyset} \leq x_{22} - x_{11}$$

These are exactly the inequality defining a (cumulative) GT-r-pattern.

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These are exactly the inequality defining a (cumulative) GT-r-pattern.

Let $\mathbb{C}[\mathrm{Gr}(k,n)]_\bullet = \{f \in \mathbb{C}[\mathrm{Gr}(k,n)] \colon f \text{ is homogeneous}\}$. Define

- $\mathrm{Cone}_{\mathrm{NO}}(G) = \mathbb{R}_{\ge 0}\text{-span}\{(\deg(f), \mathrm{val}_G(f)) \colon 0 \ne f \in \mathbb{C}[\mathrm{Gr}(k,n)]_\bullet\}$.
- $\mathrm{Cone}_W(T) = \mathbb{R}_{\ge 0}\text{-span}\{(r, v) \colon \mathrm{Trop}_T(W)(r, v) \ge 0\}$.
- $\mathrm{Cone}_{\underline{\kappa}}(T) = \mathbb{R}_{\ge 0}\text{-span}\{(\mathrm{rank}\, M, \underline{\kappa}(T, M)) \colon M \in \mathrm{CM}\, C\}$.
- $\mathrm{Cone}_{\mathrm{GV}}(T) = \mathbb{R}_{\ge 0}\text{-span}\{[T, M] \colon M \in \mathrm{CM}\, C\}$.

Remark: The four cones are the same when $T = T_{\square}$.

Theorem

Let T be a cluster tilting object and G the associated seed.

$$\mathrm{Cone}_{\underline{\kappa}}(T) = \mathrm{Cone}_W(G),$$

which can also be interpreted as

$$\mathrm{Cone}_{\mathrm{GV}}(T) = \mathrm{Cone}_W(G).$$