Classifying recollements of derived module categories for derived discrete algebras

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Plan

- Background and motivation
- Recollements
- Derived module categories of derived discrete algebras
- The main result
- Examples

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Background

- **Recollement**, introduced by Beilinson, Bernstein, and Deligne in 1980s, originates from an observation by Grothendieck on six functors in algebraic geometry.

- In representation theory, most examples of recollement are given by stratifying recollements, that is, they are of the form

$$\begin{tikzcd}
\mathcal{D}(A/AeA) \arrow[r] & \mathcal{D}(A) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathcal{D}(eAe), \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

where A is an algebra and $e \in A$ is an idempotent, such that AeA is a *stratifying ideal* (i.e. $Ae \otimes^{\mathbf{L}}_A eA \cong AeA$) of A.

Motivation

Question 1 (Xi's question)

Are all recollements of derived module categories of K-algebras stratifying up to equivalence?

- This question has a positive answer for hereditary algebras, see [Psaroudakis-Vitória'18].

- In general it is not true, as shown by Angeleri Hügel, Koenig, Liu and Yang in 2017.

C. Psaroudakis and J. Vitória, Realisation functors in tilting theory, *Math. Z*, **288** (2018), 965–1028.

L. Angeleri Hügel, S. Koenig, Q. Liu and D. Yang, Recollements and stratifying ideals, *J. Algebra*, **484** (2017), 47–65.

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Aim

Goal

- Study Xi's question within the framework of derived discrete algebras.

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Notations

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- K: field,
- A: finite dimensional algebra over K,
- $\mathrm{Mod}A$: the category of right A-modules,
- $\mathrm{mod}A$: the full subcategory consisting of all finitely generated right A-modules,
- $\mathcal{D}(A)$: unbounded derived category of $\mathrm{Mod}A$,
- $\mathcal{D}^b(\mathrm{mod}A)$: bounded derived category of $\mathrm{mod}A$,
- $\mathcal{H}^b(\mathrm{proj}A)$: the homotopy category of cochain complexes of objects in $\mathrm{proj}A$.

Recollements

Definition (Beilinson-Bernstein-Deligne 1982)

A *recollement* *of triangulated categories is a diagram*

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "i_* = i_!" description] & \mathcal{C} \arrow[l, shift right=4, "i^*"'] \arrow[l, shift left=4, "i^!"] \arrow[r, "j^! = j^*" description] & \mathcal{X} \arrow[l, shift right=4, "j_!"'] \arrow[l, shift left=4, "j_*"]
\end{tikzcd}$$

of triangulated categories and triangulated functors such that

- $(i^*, i_* = i_!, i^!), (j_!, j^! = j^*, j_*)$ *are adjoint triples;*
- i_*, j_*, $j_!$ *are fully faithful;*
- $i^! \circ j_* = 0$;
- *for each* $C \in \mathcal{C}$ *there are two induced triangles*

$$i_! i^!(C) \longrightarrow C \longrightarrow j_* j^*(C) \longrightarrow$$

$$j_! j^!(C) \longrightarrow C \longrightarrow i_* i^*(C) \longrightarrow$$

A.A. Beilinson, J. Bernstein, P. Delligne, *Faisceaux pervers, Asterisque,* Soc. Math. France **100** (1982).

Examples of recollement

Example

Let A be an algebra and let $e \in A$ be an idempotent.

- *There is a recollement*

$$\begin{tikzcd}
\mathcal{D}(B) \arrow[r, "i_*=\mathrm{inc.}"] & \mathcal{D}(A) \arrow[l, bend right, "i^*=?\otimes^{\mathbf{L}}_A B"'] \arrow[l, bend left, "i^!=\mathbb{R}\mathrm{Hom}_A(B,?)"] \arrow[r, "j^*=?\otimes^{\mathbf{L}}_A Ae"] & \mathcal{D}(eAe) \arrow[l, bend right, "j_!=?\otimes^{\mathbf{L}}_{eAe} eA"'] \arrow[l, bend left, "j_*=\mathbb{R}\mathrm{Hom}_{eAe}(Ae,?)"]
\end{tikzcd}$$

  where B is a non-positive differential graded algebra with $H^0B = A/AeA$ [Kalck-Yang 2016].

- *If moreover, AeA is stratifying, then the recollement above is stratifying, that is, B can be chosen as A/AeA, see [Cline-Parshall-Scott 1996].*

E. Cline, B. Parshall and L. Scott, Stratifying endomorphism algebras, *Mem. Amer. Math. Soc.*, **124** (1996).

M. Kalck, D. Yang, Relative singularity categories I: Auslander resolutions, *Adv. Math.*, **301** (2016), 973–1021.

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Derived discrete algebras

Definition (Vossieck 2001)

An algebra A is derived discrete provided that for every vector $\underline{d} = (d_i)_{i \in \mathbb{Z}}$ of natural numbers there are only finitely many isomorphism classes of indecomposable objects X in $\mathcal{D}^b(\mathrm{mod} A)$ of cohomology dimension vector $\left(\underline{\dim} H^i(X)\right)_{i \in \mathbb{Z}} = \underline{d}$ up to shift.

D. Vossieck, The algebras with discrete derived category, *J. Algebra*, **243** (2001), 168–176.

Example

Hereditary algebras of Dynkin type are derived discrete.

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Hereditary algebras of Dynkin type are derived discrete.

Classification of derived discrete algebras

Theorem (Bobiński-Geiß-Skowroński 2004)

Let A be a derived discrete algebra. Then A is derived equivalent to either a hereditary algebra of Dynkin type or a gentle algebra $\Lambda(r,n,m)$ with $1 \leq r \leq n$ and $m \geq 0$, given by the quiver $Q(r,n,m)$:

$$\begin{tikzcd}
 & & & 1 \arrow[r, "\alpha_1"] & \cdots \arrow[r, "\alpha_{n-r-2}"] & n-r-1 \arrow[dr, "\alpha_{n-r-1}"] & \\
-m \arrow[r, "\alpha_{-m}"] & \cdots \arrow[r, "\alpha_{-1}"] & 0 \arrow[ur, "\alpha_0"] & & & & n-r \arrow[dl, "\alpha_{n-r}"] \\
 & & & n-1 \arrow[ul, "\alpha_{n-1}"] & \cdots \arrow[l, "\alpha_{n-2}"'] & n-r+1 \arrow[l, "\alpha_{n-r+1}"'] &
\end{tikzcd}$$

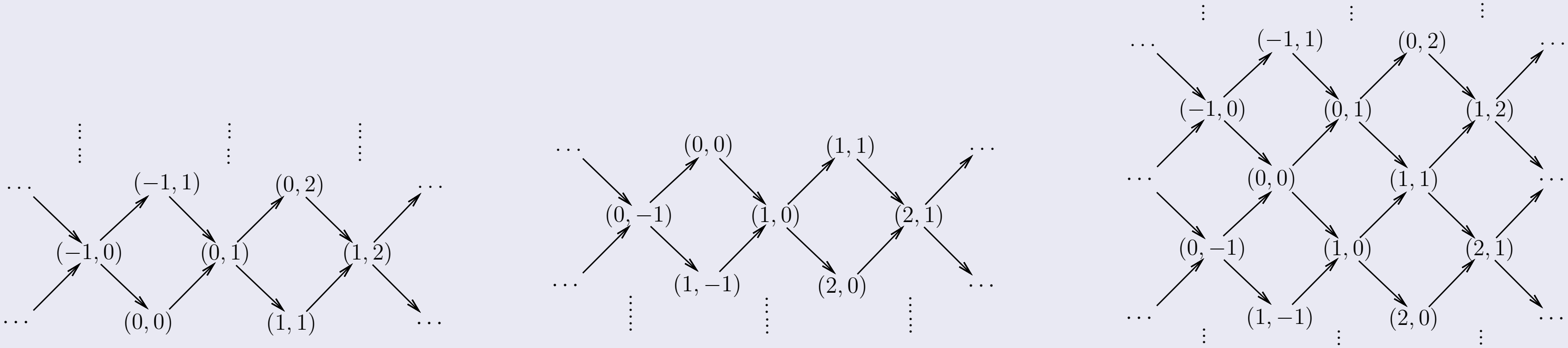
with relations $\alpha_{n-1}\alpha_0, \alpha_{n-2}\alpha_{n-1}, \ldots, \alpha_{n-r}\alpha_{n-r+1}$.

G. Bobiński, C. Geiß and A. Skowroński, Classification of discrete derived categories, *Cent. Eur. J. Math.*, **2** (2004), 19–49.

Derived categories of derived discrete algebras

AR-quiver of $\Lambda(r, n, m)$ (Bobiński-Geiß-Skowroński 2004)

The AR-quiver of $\mathcal{D}^b(\mathrm{mod}A)$ of $A = \Lambda(r, n, m)$ has precisely $3r$ components, namely, $\mathcal{X}^k$, $\mathcal{Y}^k$ and $\mathcal{Z}^k$ with $k = 0, 1, \ldots, r-1$, pictures of which can be illustrated as follows:



(a) $\mathcal{X}$ coordinates (b) $\mathcal{Y}$ coordinates (c) $\mathcal{Z}$ coordinates

Derived discrete algebras considered in this talk

From now on, A will always be the algebra $\Lambda(r, n, 0)$ with $1 \leq r < n$, that is, the bound quiver algebra given by the quiver $Q(r, n, 0)$:

$$\begin{tikzcd}
 & 1 \arrow[r, "\alpha_1"] & \cdots \arrow[r, "\alpha_{n-r-2}"] & n-r-1 \arrow[dr, "\alpha_{n-r-1}"] & \\
0 \arrow[ur, "\alpha_0"] & & & & n-r \arrow[dl, "\alpha_{n-r}"] \\
 & n-1 \arrow[ul, "\alpha_{n-1}"] & \cdots \arrow[l, "\alpha_{n-2}"] & n-r+1 \arrow[l, "\alpha_{n-r+1}"] &
\end{tikzcd}$$

with relations $\alpha_{n-1}\alpha_0, \alpha_{n-2}\alpha_{n-1}, \ldots, \alpha_{n-r}\alpha_{n-r+1}$. In this case, $\mathrm{gl.dim} A = r + 1 < \infty$.

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The main result

- An object $X \in \mathcal{D}(A)$ is an exceptional object if $\mathrm{Hom}(X, X[i \neq 0]) = 0$.
- Let $A = \Lambda(r, n, 0)$ with $1 \leq r < n$. Assume there are two finite dimensional algebras B and C, such that there is a recollement

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(B) \arrow[r, "i_*=i_!"] & \mathcal{D}(A) \arrow[l, shift right=6, "i^*"'] \arrow[l, shift left=6, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}(C). \arrow[l, shift right=6, "j_!"'] \arrow[l, shift left=6, "j_*"]
\end{tikzcd} \qquad (\mathcal{R})$$

Theorem (Xiuli Bian 2023)

Assume $j_!(C)$ or $i_*(B)$ is an indecomposable exceptional object in $\mathcal{D}(A)$. Then $(\mathcal{R})$ is equivalent to a stratifying recollement.

X. Bian, Classifying recollements of derived module categories for derived discrete algebras, *Algebr Represent Theor*, **27** (2023), 279–304.

The main result

Remark

1. *If* $X \in \mathcal{D}^b(\mathrm{mod} A)$ *is indecomposable exceptional, then* $\mathrm{End}(X) = K$ *or* $K[x]/x^2$*;*
2. *If* B *(resp.* C*) is given by* $K[x]/x^2$ *for some indecomposable exceptional object* X*, then* C *(resp.* B*) cannot be chosen as a finite dimensional algebra due to the fact below.*
3. *In the main theorem,* B *or* C *is* K*.*
4. *In the main theorem,* $i_*(B)$ *or* $j_!(C)$ *is indecomposable object in* $\mathcal{Y}$ *or* $\mathcal{Z}$ *component.*

Fact (Angeleri Hügel-Koenig-Liu-Yang 2017)

If A*,* B *and* C *are finite dimensional with recollement* $(\mathcal{R})$*, then* $\mathrm{gl.dim} A < \infty$ *if and only if* $\mathrm{gl.dim} B < \infty$ *and* $\mathrm{gl.dim} C < \infty$*.*

L. Angeleri Hügel, S. Koenig, Q. Liu and D. Yang, Ladders and simplicity of derived module categories, *J. Algebra*, **472** (2017), 15–66.

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Example

Example

Consider the algebra $A = \Lambda(2,4,0)$

$$\begin{tikzcd}
 & 1 \arrow[dr, "\alpha_1"] & \\
0 \arrow[ur, "\alpha_0"] & & 2 \arrow[dl, "\alpha_2"] \\
 & 3 \arrow[ul, "\alpha_3"] &
\end{tikzcd}$$

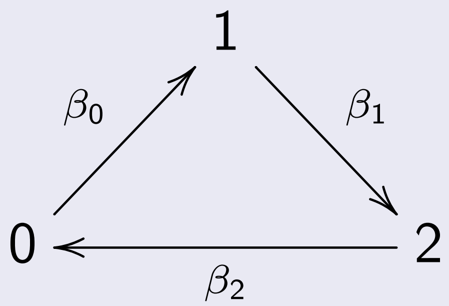
with relations $\alpha_3\alpha_0$, $\alpha_2\alpha_3$.

Example

- P_0 in $\mathcal{X}$ generates a recollement

$$\begin{tikzcd}
\mathcal{D}(B) \arrow[r, "i_*=i_!" description] & \mathcal{D}(A) \arrow[l, bend right, "i^*" description] \arrow[l, bend left, "i^!" description] \arrow[r, "j^!=j^*" description] & \mathcal{D}(C) \arrow[l, bend right, "j_!" description] \arrow[l, bend left, "j_*" description]
\end{tikzcd} ,$$

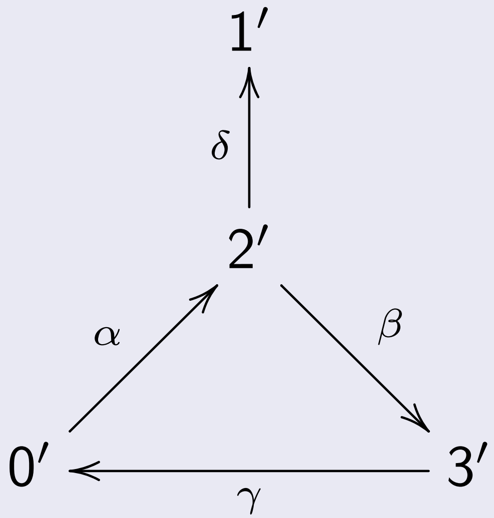
where $C = \mathrm{End}_A(P_0) \simeq K$, B is a dg-algebra which is derived equivalent to the graded bound quiver algebra given by:



with relation $\beta_2\beta_0$, and $deg\,\beta_0 = -1$, $deg\,\beta_1 = deg\,\beta_2 = 0$. This is not a stratifying recollement (because B is not derived equivalent to a finite dimensional algebra).

Example

- Consider the recollement generated by S_1 (which is in $\mathcal{Y}$-component):
  A is derived equivalent to the algebra A'

  with relation $\alpha\delta$, $\gamma\alpha$, $\beta\gamma$, sends S_1 to P_1'.

Example

Using A', we show that the recollement generated by S_1 is equivalent to

$$\begin{tikzcd}
\mathcal{D}(B) \arrow[r, "i_*=i_!"] & \mathcal{D}(A) \arrow[l, shift right=4ex, "i^*"'] \arrow[l, shift left=4ex, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}(C=K) \arrow[l, shift right=4ex, "j_!"'] \arrow[l, shift left=4ex, "j_*"]
\end{tikzcd} ,$$

with $B \simeq \Lambda(2,3,0)$. This is a stratifying recollement.
Moreover, by extending this recollement one-step downwards, we can also obtain a new recollement which is stratifying as well.

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Moreover, by extending this recollement one-step downwards, we can also obtain a new recollement which is stratifying as well.

The main result

- P_1 in $\mathcal{Z}$ generates a recollement

$$\begin{tikzcd}
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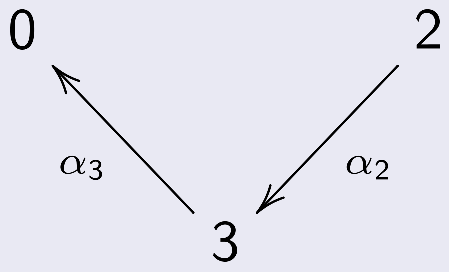
\end{tikzcd} ,$$

with $C = \mathrm{End}_A(P_1) \simeq K$ and B is given by



with relations $\alpha_2\alpha_3$. This is a stratifying recollement. Moreover, by extending this recollement one-step downwards, we can also obtain a new recollement which is stratifying as well.

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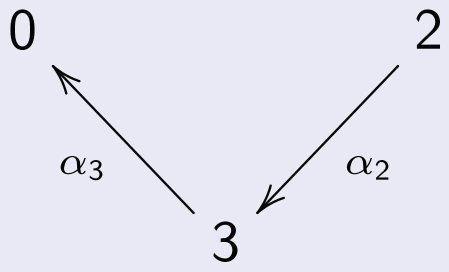

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Thank you!