Auslander-Reiten translations in the monomorphism categories of exact categories

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- Operator theory: Classify invariant subspaces of a linear nilpotent operator (Ringel-Schmidmeier [RS08b]).

- For a category $\mathcal{C}$, denote by $\mathcal{H}(\mathcal{C})$ its morphism category
  Objects: morphisms $f : A \to B$ in $\mathcal{C}$.
  Morphisms: $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} : f_1 \Rightarrow f_2$ is a commutative diagram:

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- If $\mathcal{C}$ is exact, then the monomorphism category is

$$\mathcal{S}(\mathcal{C}) = \{f \in \mathcal{H}(\mathcal{C}) | f \text{ is an inflation (monomorphism)}\}.$$

  The epimorphism category is

$$\mathcal{F}(\mathcal{C}) = \{f \in \mathcal{H}(\mathcal{C}) | f \text{ is an deflation (epimorphism)}\}.$$

- Monomorphism categories are related with many subjects: Gorenstein-projective modules, cotorsion theories, singularity categories, weighted projective lines, CY-categories, representations of tensor algebras, functor categories, monads and EM-categories. [LZ13, LZ17, Zha11, KLM13, GKKP22, GKKP23]

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- And have applications in other fields of mathematics:
  Operator theory, Differential equations,
  Dynamical systems, Control theory,
  Data Science and Theoretical mechanics.

Motivations

Theorem ([RS08a])

Let Λ be an artin algebra. The category $\mathcal{S}(\Lambda)$ and $\mathcal{F}(\Lambda)$ are functorially finite in $\mathcal{H}(\Lambda)$, hence having almost split sequences. The AR-translations in $\mathcal{S}(\Lambda)$ and $\mathcal{F}(\Lambda)$ satisfy

$$\tau_{\mathcal{S}}(f) \cong \mathbf{Mimo}\, \tau_\Lambda \operatorname{Coker}(f), \tau_{\mathcal{F}}(g) \cong \operatorname{Coker} \mathbf{Mimo}\, \tau_\Lambda(g)$$

where $\mathbf{Mimo}(h : X \to Y) = (X \overset{[h,e]^T}{\hookrightarrow} Y \oplus I \ker f)$ *is a minimal right $\mathcal{S}(\Lambda)$-approximation of h (Here, $e : X \to I \ker f$ is an extension of the injective envelope of $\ker f$).*

(Note: $\mathcal{H}(\Lambda) \cong \Lambda \otimes kA_2 - \text{mod}$ is a module category.)

Theorem (Luo-Zhu arXiv:2408.01359v1)

Let Λ be an artin algebra. Let $\mathcal{C}$ be a functorially finite exact subcategory of Λ-mod with enough projective and injective objects. Then $\mathcal{S}(\mathcal{C})$ and $\mathcal{F}(\mathcal{C})$ are functorially finite in $\mathcal{H}(\mathcal{C})$ and $\mathcal{H}(\Lambda)$. The AR-translations in $\mathcal{S}(\mathcal{C})$ and $\mathcal{F}(\mathcal{C})$ satisfy:

$$\tau_{\mathcal{S}}(f) \cong \mathbf{Mimo}\,\tau_{\mathcal{C}}\operatorname{Coker}(f), \tau_{\mathcal{F}}(g) \cong \operatorname{Coker}\mathbf{Mimo}\,\tau_{\mathcal{C}}(g).$$

Transfering AR-sequences

$$\mathcal{H}(\Lambda) \xrightarrow{\quad\text{Coker}\quad} \mathcal{F}(\Lambda)$$

$$\begin{array}{ccc} & & \underline{\mathcal{F}(\mathcal{C})} \\ & & \Big\downarrow \underline{?} \\ (\underline{\mathcal{C}}\text{-mod}) & \xleftarrow{\quad \Theta = \text{Coker Hom}(-,?) \quad} & \mathcal{H}(\underline{\mathcal{C}}) \end{array}$$

Figure: Ringel-Schmidmeier's approach, new approach

AR-sequences in $\mathcal{S}(\mathcal{C})$ and $\mathcal{F}(\mathcal{C})$

If $\mathcal{C}$ is an exact category, then $\mathcal{H}(\mathcal{C})$ has a natural exact structure inherited from $\mathcal{C}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C_1 & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A_2 & \xrightarrow{\alpha_2} & B_2 & \xrightarrow{\beta_2} & C_2 & \longrightarrow & 0 \end{array} \qquad (*)$$

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Furthermore, if $\mathcal{C}$ is an extension-closed dualizing subvariety of $\Lambda-$mod, then $\mathcal{H}(\mathcal{C})$, $\mathcal{S}(\mathcal{C})$ and $\mathcal{F}(\mathcal{C})$ have AR-sequences.

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Lemma ([RS08a, HE23])

Let (*) be an almost split sequence in $\mathcal{H}(\mathcal{C})$ or $\mathcal{S}(\mathcal{C})$ or $\mathcal{F}(\mathcal{C})$.
(1) If h is not an isomorphism, then the top row splits.
(2) If f is not an isomorphism, then the bottom row splits.

An exact sequence in $\mathcal{H}(\mathcal{C})$ is called **cw-exact** if both rows are split exact.

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Non-cw-exact AR-sequences are of the forms:

$$0 \longrightarrow \begin{pmatrix} A \\ A \end{pmatrix}_1 \xrightarrow{\begin{pmatrix} 1 \\ f \end{pmatrix}} \begin{pmatrix} A \\ B \end{pmatrix}_f \xrightarrow{\begin{pmatrix} 0 \\ g \end{pmatrix}} \begin{pmatrix} 0 \\ C \end{pmatrix}_0 \longrightarrow 0,$$

$$0 \longrightarrow \begin{pmatrix} A \\ 0 \end{pmatrix}_0 \xrightarrow{\begin{pmatrix} f \\ 0 \end{pmatrix}} \begin{pmatrix} B \\ C \end{pmatrix}_g \xrightarrow{\begin{pmatrix} g \\ 1 \end{pmatrix}} \begin{pmatrix} C \\ C \end{pmatrix}_1 \longrightarrow 0,$$

where $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an almost split sequence in $\mathcal{C}$. Similarly for $\mathcal{S}(\mathcal{C})$.

Key Technique

[HE23] The functor

$$\begin{aligned}\Theta : \mathcal{H}(\mathcal{C}) &\to \mathcal{C}-\mathrm{mod}\\ f &\mapsto \mathrm{Coker}\,\mathrm{Hom}_{\mathcal{C}}(-,f)\end{aligned}$$

sends cw-exact AR sequences to AR sequences.

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Remark

Morphism category=Functor category $+\ \varepsilon$.

[AR74] Let $\mathcal{C}$ be a dualizing variety. Given a finitely presented functor $(-,B) \xrightarrow{(-,g)} (-,C) \to F \to 0$, the transpose of F is given by the

$$(C,-) \xrightarrow{(g,-)} (B,-) \to TrF \to 0.$$

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Hence it fits into a diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & DTrF & \longrightarrow & D(B,-) & \xrightarrow{D(g,-)} & D(C,-) \\
 & & & & \downarrow \wr & & \downarrow \wr \\
(-,\tau C \oplus I) & \xrightarrow{(-,q)} & (-,D) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{C}}(-,\tau B) & \xrightarrow{\operatorname{Ext}^1(-,\tau_{\mathcal{C}} g)} & \operatorname{Ext}^1(-,\tau C)
\end{array}$$

where $0 \to \tau B \xrightarrow{\textbf{Mimo}\,\tau_{\mathcal{C}} g} \tau C \oplus I \xrightarrow{q} D \to 0$ is exact.

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