G-dimensions for DG-modules over commutative DG-rings

joint with Jiangsheng Hu, Rongmin Zhu

ShangHai RCRA 21

2024.8.8

Outline

A: ring or DG-ring
$\mathrm{D}(A)$: the der.cat. over A

$\mathrm{D_f}(A)$: the full tri.subcat.of objects X s.t. $\mathrm{H}^n(X)$ is f.g.

$\mathrm{D}^+(A)$: the full tri.subcat.of objects X s.t. $\mathrm{H}^n(X) = 0$ for $n \ll 0$,
$\mathrm{D}^+_\mathrm{f}(A) = \mathrm{D}^+(A) \cap \mathrm{D_f}(A)$

$\mathrm{D}^-(A)$: the full tri.subcat.of objects X s.t. $\mathrm{H}^n(X) = 0$ for $n \gg 0$,
$\mathrm{D}^-_\mathrm{f}(A) = \mathrm{D}^-(A) \cap \mathrm{D_f}(A)$

$\mathrm{D^b}(A) = \mathrm{D}^-(A) \cap \mathrm{D}^+(A)$, $\mathrm{D^b_f}(A) = \mathrm{D^b}(A) \cap \mathrm{D_f}(A)$

The projective dimension is a most important invariant for modules; this is illustrated by the next two results.

Regularity Theorem for modules

(Serre, Auslander, Buchsbaum) TFAE

• The local ring $(A, \mathfrak{m})$ is regular (the maximal ideal $\mathfrak{m}$ can be generated by $\dim A$ elements);

• $k = A/\mathfrak{m}$ has finite projective dimension;

• All f.g. A-modules has finite projective dimension;

Auslander-Buchsbaum Formula

A: local ring, M: f.g. A-module. If $\mathrm{projdim}_A M < \infty$, then

$$\mathrm{projdim}_A M = \mathrm{depth} A - \mathrm{depth}_A M.$$

The projective dimension is a most important invariant for modules; this is illustrated by the next two results.

Regularity Theorem for modules

(Serre, Auslander, Buchsbaum) TFAE

• The local ring $(A, \mathfrak{m})$ is regular (the maximal ideal $\mathfrak{m}$ can be generated by $\dim A$ elements);

• $k = A/\mathfrak{m}$ has finite projective dimension;

• All f.g. A-modules has finite projective dimension;

Auslander-Buchsbaum Formula

A: local ring, M: f.g. A-module. If $\operatorname{projdim}_A M < \infty$, then

$$\operatorname{projdim}_A M = \operatorname{depth} A - \operatorname{depth}_A M.$$

The projective dimension is a most important invariant for modules; this is illustrated by the next two results.

Regularity Theorem for modules

(Serre, Auslander, Buchsbaum) TFAE

- The local ring $(A, \mathfrak{m})$ is regular (the maximal ideal $\mathfrak{m}$ can be generated by $\dim A$ elements);
- $k = A/\mathfrak{m}$ has finite projective dimension;
- All f.g. A-modules has finite projective dimension;

Auslander-Buchsbaum Formula

A: local ring, M: f.g. A-module. If $\mathrm{projdim}_A M < \infty$, then

$$\mathrm{projdim}_A M = \mathrm{depth} A - \mathrm{depth}_A M.$$

Regularity Theorem for complexes

TFAE

- The local ring $(A, \mathfrak{m})$ is regular;
- $k = A/\mathfrak{m}$ has finite projective dimension;
- All A-complexes in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ has finite projective dimension;

Auslander-Buchsbaum Formula for complexes

A: local ring, $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. If $\mathrm{projdim}_A M < \infty$, then

$$\mathrm{projdim}_A M = \mathrm{depth} A - \mathrm{depth}_A M.$$

Homological methods found their way into commutative algebra.

[1] M. Auslander, D.A. Buchsbaum, Homological dimension in Noetherian rings, *Proc. Nat. Acad. Sci.* 42 (1956) 36–38.

[2] J.-P. Serre, *Surla dimension homologique des anneaux et des modules noethdriens*, Proceedings of the international symposium on algebraic number theory, Tokyo § Nikko, 1955 (Tokyo), Science Council of Japan, 1956, 175–189.

Regularity Theorem for complexes

TFAE

- The local ring $(A, \mathfrak{m})$ is regular;
- $k = A/\mathfrak{m}$ has finite projective dimension;
- All A-complexes in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ has finite projective dimension;

Auslander-Buchsbaum Formula for complexes

A: local ring, $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. If $\mathrm{projdim}_A M < \infty$, then

$$\mathrm{projdim}_A M = \mathrm{depth} A - \mathrm{depth}_A M.$$

Homological methods found their way into commutative algebra.

[1] M. Auslander, D.A. Buchsbaum, Homological dimension in Noetherian rings, *Proc. Nat. Acad. Sci.* 42 (1956) 36–38.

[2] J.-P. Serre, *Surla dimension homologique des anneaux et des modules noethdriens*, Proceedings of the international symposium on algebraic number theory, Tokyo § Nikko, 1955 (Tokyo), Science Council of Japan, 1956, 175–189.

Regularity Theorem for complexes

TFAE

- The local ring $(A, \mathfrak{m})$ is regular;
- $k = A/\mathfrak{m}$ has finite projective dimension;
- All A-complexes in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ has finite projective dimension;

Auslander-Buchsbaum Formula for complexes

A: local ring, $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. If $\mathrm{projdim}_A M < \infty$, then

$$\mathrm{projdim}_A M = \mathrm{depth} A - \mathrm{depth}_A M.$$

Homological methods found their way into commutative algebra.

[1] M. Auslander, D.A. Buchsbaum, Homological dimension in Noetherian rings, *Proc. Nat. Acad. Sci.* 42 (1956) 36–38.

[2] J.-P. Serre, *Surla dimension homologique des anneaux et des modules noethdriens*, Proceedings of the international symposium on algebraic number theory, Tokyo § Nikko, 1955 (Tokyo), Science Council of Japan, 1956, 175–189.

The G-dimension for f.g. module over a commutative noetherian ring was introduced by Auslander in [3], and was developed deeply by Auslander and Bridger in [4].

A: comm. noetherian ring. A f.g. A-module M is in $G(A)$ iff
(i) $\mathrm{Ext}_A^m(M,A) = 0$ for $m \neq 0$;
(ii) $\mathrm{Ext}_A^m(\mathrm{Hom}_A(M,A),A) = 0$ for $m \neq 0$;
(iii) The biduality map $\delta_M : M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M,A),A)$ is an iso.

All conditions in Definition are necessary.

[3] M. Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Séminaire d'algèbre commutative dirigé par P. Samuel, Secrétariat mathématique, Paris, 1967.

[4] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. vol. 94, 1969.

The G-dimension for f.g. module over a commutative noetherian ring was introduced by Auslander in [3], and was developed deeply by Auslander and Bridger in [4].

A: comm. noetherian ring. A f.g. A-module M is in $G(A)$ iff
(i) $\mathrm{Ext}_A^m(M, A) = 0$ for $m \neq 0$;
(ii) $\mathrm{Ext}_A^m(\mathrm{Hom}_A(M, A), A) = 0$ for $m \neq 0$;
(iii) The biduality map $\delta_M : M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A)$ is an iso.

All conditions in Definition are necessary.

[3] M. Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Séminaire d'algèbre commutative dirigé par P. Samuel, Secrétariat mathématique, Paris, 1967.

[4] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. vol. 94, 1969.

The G-dimension for f.g. module over a commutative noetherian ring was introduced by Auslander in [3], and was developed deeply by Auslander and Bridger in [4].

A: comm. noetherian ring. A f.g. A-module M is in $G(A)$ iff
(i) $\mathrm{Ext}_A^m(M, A) = 0$ for $m \neq 0$;
(ii) $\mathrm{Ext}_A^m(\mathrm{Hom}_A(M, A), A) = 0$ for $m \neq 0$;
(iii) The biduality map $\delta_M : M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A)$ is an iso.

All conditions in Definition are necessary.

[3] M. Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Séminaire d'algèbre commutative dirigé par P. Samuel, Secrétariat mathématique, Paris, 1967.

[4] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. vol. 94, 1969.

$$\{\text{f.g. projective } A\text{-modules}\} \subsetneq G(A)$$

k: field, $A = k[[x]]/(x^2)$: local and self-injective (0-Gorenstein ring) $\Rightarrow k \in G(A)$, but k is not projective because A is not regular.

A f.g. A-module M is said to have finite G-dimension if it has a $G(A)$-resolution of finite length, i.e. $\exists$ exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

with each $G_i \in G(A)$. G-dimension is a refinement of projective dimension for f.g. modules.

$$\{\text{f.g. projective } A\text{-modules}\} \subsetneq G(A)$$

k: field, $A = k[[x]]/(x^2)$: local and self-injective (0-Gorenstein ring) $\Rightarrow k \in G(A)$, but k is not projective because A is not regular.

A f.g. A-module M is said to have finite G-dimension if it has a $G(A)$-resolution of finite length, i.e. $\exists$ exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

with each $G_i \in G(A)$. G-dimension is a refinement of projective dimension for f.g. modules.

The G-dimension shares many of the nice properties of the projective dimension.

Gorenstein Theorem for modules

TFAE

- The local ring $(A, \mathfrak{m})$ is Gorenstein ($\mathrm{injdim}_A A < \infty$);
- $k = A/\mathfrak{m}$ has finite G-dimension;
- All f.g. A-modules has finite G-dimension.

---

[5] L.W. Christensen, *Gorenstein Dimensions*, Lecture Notes in Math., vol. 1747, Springer-Verlag, Berlin, 2000.

The G-dimension shares many of the nice properties of the projective dimension.

Gorenstein Theorem for modules

TFAE

- The local ring $(A, \mathfrak{m})$ is Gorenstein $(\mathrm{injdim}_A A < \infty)$;
- $k = A/\mathfrak{m}$ has finite G-dimension;
- All f.g. A-modules has finite G-dimension.

---

[5] L.W. Christensen, *Gorenstein Dimensions*, Lecture Notes in Math., vol. 1747, Springer-Verlag, Berlin, 2000.

G-dimension has played an important role in singularity theory, cohomology theory of commutative rings and representation theory of Artin algebras.

[6] M. Auslander and I. Reiten, *Applications of contravariantly finite subcategories*, Adv. Math. 86 (1991) 111–152.

[7] L.L. Avramov and A. Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. 85 (2002) 393–440.

[8] R.-O. Buchweitz, *Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings*, with appendices by Luchezar L. Avramov, Benjamin Briggs, Srikanth B. Iyengar, and Janina C. Letz, Math. Surveys and Monographs 262, Amer. Math. Soc. 2021.

[9] C.M. Ringel and P. Zhang, *Representations of quivers over the algebra of dual numbers*, J. Algebra 475 (2017) 327–360.

[10] Y. Yoshino, *Cohen-Macaulay Modules Over Cohen-Macaulay Rings*, London Math. Soc. Lecture Note Ser., vol. 146, Cambridge Univ. Press, Cambridge, 1990.

Yassemi [11] studied G-dimension for complexes through a consistent use of the RHom-functor of complexes and the related category of reflexive complexes.

A: comm. noetherian local ring. An A-complex X is said to be reflexive iff

- $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$;
- $\mathrm{RHom}_A(X, A) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$;
- X represents $\mathrm{RHom}_A(\mathrm{RHom}_A(X, A), A)$ canonically.

[11] S. Yassemi, *G-dimension*, Math. Scand. 77 (1995) 161–174.

Christensen [5] gave the definition of G-dimension for complexes in terms of resolutions.

$$\text{G-dim}_A X = \inf\{\sup\{l \in \mathbb{Z} \mid G^{-l} \neq 0\} \mid X \simeq G \in \mathrm{C}^{G(A)}_{(\sqsupset)}(A)\}.$$

The two definitions are equivalent, and they are both rooted in a result — due to Foxby.
An A-complex X is reflexive iff X has finite G-dimension and

$$\text{G-dim}_A X = \sup \mathrm{RHom}_A(X, A).$$

---

[5] L.W. Christensen, *Gorenstein Dimensions*, Lecture Notes in Math., vol. 1747, Springer-Verlag, Berlin, 2000.

Gorenstein Theorem for complexes

TFAE

- The local ring $(A, \mathfrak{m})$ is Gorenstein;
- $k = A/\mathfrak{m}$ has finite G-dimension;
- All A-complexes in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ has finite G-dimension.

Despite the great success of the G-dimension in comm. noetherian rings, until now it was completely missing from higher algebra. For example, non-positive comm. DG-rings.

DG-ring $A = \bigoplus_{i\in\mathbb{Z}} A^i$: $\mathbb{Z}$-graded ring with map $d_A : A \to A$ s.t. $d_A \circ d_A = 0$, and satisfies the Leibniz rule

$$d_A(a \cdot b) = d_A(a) \cdot b + (-1)^{|a|} a \cdot d_A(b),\ \forall\ a, b \in A.$$

commutative DG-ring A: $b \cdot a = (-1)^{|a|\cdot|b|} a \cdot b,\ \forall\ a, b \in A$, and $a^2 = 0$ if $|a|$ is odd.
non-positive DG-ring A: $A^i = 0$ for all $i > 0$.
noetherian DG-ring A: $\mathrm{H}^0(A)$ is noetherian, $\mathrm{H}^i(A)$ is a f.g. $\mathrm{H}^0(A)$-module for $i < 0$.
local noetherian DG-ring $(A, \bar{\mathfrak{m}}, \bar{\kappa})$: $(\mathrm{H}^0(A), \bar{\mathfrak{m}}, \bar{\kappa})$ is local and A is noetherian.

Despite the great success of the G-dimension in comm. noetherian rings, until now it was completely missing from higher algebra. For example, non-positive comm. DG-rings.

DG-ring $A = \bigoplus_{i\in\mathbb{Z}} A^i$: $\mathbb{Z}$-graded ring with map $d_A : A \to A$ s.t. $d_A \circ d_A = 0$, and satisfies the Leibniz rule

$$d_A(a \cdot b) = d_A(a) \cdot b + (-1)^{|a|} a \cdot d_A(b), \forall\, a, b \in A.$$

commutative DG-ring A: $b \cdot a = (-1)^{|a|\cdot|b|} a \cdot b$, $\forall\, a, b \in A$, and $a^2 = 0$ if $|a|$ is odd.
non-positive DG-ring A: $A^i = 0$ for all $i > 0$.
noetherian DG-ring A: $\mathrm{H}^0(A)$ is noetherian, $\mathrm{H}^i(A)$ is a f.g. $\mathrm{H}^0(A)$-module for $i < 0$.
local noetherian DG-ring $(A, \bar{\mathfrak{m}}, \bar{\kappa})$: $(\mathrm{H}^0(A), \bar{\mathfrak{m}}, \bar{\kappa})$ is local and A is noetherian.

Despite the great success of the G-dimension in comm. noetherian rings, until now it was completely missing from higher algebra. For example, non-positive comm. DG-rings.

DG-ring $A = \bigoplus_{i\in\mathbb{Z}} A^i$: $\mathbb{Z}$-graded ring with map $d_A : A \to A$ s.t. $d_A \circ d_A = 0$, and satisfies the Leibniz rule

$$d_A(a\cdot b) = d_A(a)\cdot b + (-1)^{|a|} a\cdot d_A(b), \forall\, a, b \in A.$$

commutative DG-ring A: $b\cdot a = (-1)^{|a|\cdot|b|} a\cdot b$, $\forall\, a, b \in A$, and $a^2 = 0$ if $|a|$ is odd.
non-positive DG-ring A: $A^i = 0$ for all $i > 0$.
noetherian DG-ring A: $\mathrm{H}^0(A)$ is noetherian, $\mathrm{H}^i(A)$ is a f.g. $\mathrm{H}^0(A)$-module for $i < 0$.
local noetherian DG-ring $(A, \bar{\mathfrak{m}}, \bar{\kappa})$: $(\mathrm{H}^0(A), \bar{\mathfrak{m}}, \bar{\kappa})$ is local and A is noetherian.

DG-module: a graded A-module (X, d_X) with the differential $d_X : X \to X$ satisfies the Leibniz rule

$$d_X(ax) = d_A(a)x + (-1)^{|a|}a \cdot d_X(x), \forall\ a \in A,\ x \in X.$$

The infimum, supremum and amplitude of a DG-module X

$$\inf X := \inf\{n \in \mathbb{Z} | \mathrm{H}^n(X) \neq 0\},$$
$$\sup X := \sup\{n \in \mathbb{Z} | \mathrm{H}^n(X) \neq 0\},$$
$$\mathrm{amp} X := \sup X - \inf X.$$

We refer the reader to [12] for more details about DG rings and their derived categories.

[12] A. Yekutieli, *Derived Categories*, Cambridge Studies in Advanced Mathematics, Cambridge: Cambridge University Press, 2019.

DG-module: a graded A-module (X, d_X) with the differential $d_X : X \to X$ satisfies the Leibniz rule

$$d_X(ax) = d_A(a)x + (-1)^{|a|}a \cdot d_X(x), \forall\ a \in A,\ x \in X.$$

The infimum, supremum and amplitude of a DG-module X

$$\mathrm{inf}X := \inf\{n \in \mathbb{Z} | \mathrm{H}^n(X) \neq 0\},$$
$$\mathrm{sup}X := \sup\{n \in \mathbb{Z} | \mathrm{H}^n(X) \neq 0\},$$
$$\mathrm{amp}X := \mathrm{sup}X - \mathrm{inf}X.$$

We refer the reader to [12] for more details about DG rings and their derived categories.

[12] A. Yekutieli, *Derived Categories*, Cambridge Studies in Advanced Mathematics, Cambridge: Cambridge University Press, 2019.

DG-module: a graded A-module (X, d_X) with the differential $d_X : X \to X$ satisfies the Leibniz rule

$$d_X(ax) = d_A(a)x + (-1)^{|a|} a \cdot d_X(x), \forall\, a \in A,\ x \in X.$$

The infimum, supremum and amplitude of a DG-module X

$$\mathrm{inf}X := \inf\{n \in \mathbb{Z} \mid \mathrm{H}^n(X) \neq 0\},$$
$$\mathrm{sup}X := \sup\{n \in \mathbb{Z} \mid \mathrm{H}^n(X) \neq 0\},$$
$$\mathrm{amp}X := \mathrm{sup}X - \mathrm{inf}X.$$

We refer the reader to [12] for more details about DG rings and their derived categories.

---

[12] A. Yekutieli, *Derived Categories*, Cambridge Studies in Advanced Mathematics, Cambridge: Cambridge University Press, 2019.

Example

1. k: field, A: the ring $A = k[y,z]/(y^2, yz, z^2)$ with $d_A = 0$, where y and z are indeterminates over k. If $|y| = -1 = |z|$, then A concentrates in 0,-1 with $0 < \mathrm{amp}A < \infty$.
2. A: graded ring over $k = \mathbb{Z}/(4)$ with A^i a free k-module on a basis element x^i for $i \in \mathbb{Z}$, and $x^i x^j = x^{i+j}$ for $i, j \in \mathbb{Z}$. Then A is a DG-ring with $\mathrm{amp}A = \infty$.

Example

1. k: field, A: the ring $A = k[y,z]/(y^2, yz, z^2)$ with $d_A = 0$, where y and z are indeterminates over k. If $|y| = -1 = |z|$, then A concentrates in 0,-1 with $0 < \mathrm{amp} A < \infty$.
2. A: graded ring over $k = \mathbb{Z}/(4)$ with A^i a free k-module on a basis element x^i for $i \in \mathbb{Z}$, and $x^i x^j = x^{i+j}$ for $i, j \in \mathbb{Z}$. Then A is a DG-ring with $\mathrm{amp} A = \infty$.

Example

1. k: field, A: the ring $A = k[y,z]/(y^2, yz, z^2)$ with $d_A = 0$, where y and z are indeterminates over k. If $|y| = -1 = |z|$, then A concentrates in 0,-1 with $0 < \mathrm{amp}A < \infty$.
2. A: graded ring over $k = \mathbb{Z}/(4)$ with A^i a free k-module on a basis element x^i for $i \in \mathbb{Z}$, and $x^i x^j = x^{i+j}$ for $i,j \in \mathbb{Z}$. Then A is a DG-ring with $\mathrm{amp}A = \infty$.

G-dimension for DG-modules

Let A be a noetherian DG-ring with $\mathrm{amp}A < \infty$.

1. $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ is said to be *reflexive* if $\mathrm{RHom}_A(X,A) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ and the map $X \to \mathrm{RHom}_A(\mathrm{RHom}_A(X,A),A)$ is isomorphic in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. We denote by $\mathcal{R}(A)$ for the full subcategory of $\mathrm{D}(A)$ consisting of reflexive DG-modules.
2. For $X \in \mathcal{R}(A)$, define the *G-dimension* of X by the formula

$$\mathrm{G}\text{-}\mathrm{dim}_A X = \mathrm{sup}\mathrm{RHom}_A(X,A).$$

   If X is not reflexive, we say that $\mathrm{G}\text{-}\mathrm{dim}_A X = \infty$.
3. $X \in \mathcal{R}(A)$ is said to be in the *G-class* $\mathcal{G}$ if either $\mathrm{G}\text{-}\mathrm{dim}_A X = -\mathrm{sup}X$, or $X \simeq 0$, and denote by $\mathcal{G}_0$ the full subcategory of $\mathcal{G}$ consisting of objects G such that either $\mathrm{amp}G \geq \mathrm{amp}A$ and $\mathrm{G}\text{-}\mathrm{dim}_A G = -\mathrm{sup}G = 0$, or $G \simeq 0$.

Remark

1. A: ordinary ring, X: A-module $\Longrightarrow$ $\text{G-dim}_A X$ coincides with the usual definition of the G-dimension for modules, and $\mathcal{G} = \mathcal{G}_0$ is exactly the class $G(A)$.
2. A: ordinary ring, X: A-complex $\Longrightarrow$ $\text{G-dim}_A X$ is just the definition of the G-dimension for complexes, and $\text{G-dim}_A X \geq -\text{sup} X$.
3. One major difference from the case of rings is that $\text{G-dim}_A X \geq -\text{sup} X$ need not hold in the DG-setting.

Example (Dong Yang)

k: field, $A = k[x]/(x^2)$: DG k-algebra with $\deg(x) = -1$, $S = A/(x)$. Then S has a resolution

$$\cdots \to A[2] \xrightarrow{x} A[1] \xrightarrow{x} A \to S \to 0.$$

Its total complex F is a minimal semi-free resolution of S, the DG-module $\mathrm{Hom}(F, A)$ is the total complex of

$$0 \to A \xrightarrow{x} A[-1] \xrightarrow{x} A[-2] \to \cdots .$$

Then $\mathrm{RHom}(S, A) \simeq \mathrm{Hom}(F, A) \simeq S[1]$ and $S \in \mathcal{R}(A)$. So in this case, $\sup S = 0$, but $\text{G-dim}_A S = \sup \mathrm{RHom}(S, A) = -1$.

sppj resolution (Minamoto, 2021)

For any $0 \not\simeq X \in \mathrm{D}^{+}(A)$,

- *sppj morphism* $f : P \to X$ is a morphism in $\mathrm{D}(A)$ such that $P \in \mathrm{Add}A[-\mathrm{sup}X]$ and the morphism $\mathrm{H}^{\mathrm{sup}X}(f)$ is surjective.
- *sppj resolution* $P_\bullet$ of X is a sequence of exact triangles

$$X_{i+1} \xrightarrow{g_{i+1}} P_i \xrightarrow{f_i} X_i \rightsquigarrow$$

s.t. f_i is a sppj morphism for $i \geq 0$ with $X_0 := X$.

- A: noetherian, $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) \Rightarrow P_i \in \mathcal{P} := \mathrm{add}A$.
- For full subcategories $\mathcal{X}, \mathcal{Y} \subseteq \mathrm{D}(A)$,

$$\mathcal{X} * \mathcal{Y} = \{Z \in \mathrm{D}(A) \,|\, \exists\, X \to Z \to Y \rightsquigarrow,\ X \in \mathcal{X}, Y \in \mathcal{Y}\}.$$

[13] H. Minamoto, *Resolutions and homological dimensions of DG-modules*, Israel J. Math. 245 (2021) 409–454.

Theorem (HYZ)

$(A, \bar{\mathfrak{m}}, \bar{k})$: comm. noetherian local DG-ring with $\operatorname{amp} A < \infty$.

(1) If $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\operatorname{amp} X \geq \operatorname{amp} A$ and $n \geq 0$, TFAE:
(i) $\text{G-dim}_A X \leq n - \sup X$;
(ii) A has a sppj resolution $P_\bullet$ of X such that $X_n \in \mathcal{G}_0[-\sup X_n]$;
(iii) $X \in \mathcal{P}[-\sup X] * \cdots * \mathcal{P}[-\sup X + n - 1] * \mathcal{G}_0[-\sup X + n]$;
(iv) $X \in \mathcal{G}_0[-\sup X] * \mathcal{P}[-\sup X + 1] * \cdots * \mathcal{P}[-\sup X + n]$.

(2) If $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\operatorname{amp} X < \operatorname{amp} A$ and $n \geq 0$, TFAE:
(i) $\text{G-dim}_A X \leq n + \inf A - \inf X$;
(ii) $X \oplus X[\operatorname{amp} A - \operatorname{amp} X] \in \mathcal{P}[-\sup X] * \cdots * \mathcal{P}[-\sup X + n - 1] * \mathcal{G}_0[-\sup X + n]$.

(3) TFAE:
(i) A is local Gorenstein ($\operatorname{amp} A < \infty$ and $\operatorname{injdim}_A A < \infty$);
(ii) $\text{G-dim}_A \bar{k} < \infty$;
(iii) $\text{G-dim}_A X < \infty$ for any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.

Theorem (HYZ)

$(A, \bar{\mathfrak{m}}, \bar{k})$: comm. noetherian local DG-ring with $\mathrm{amp}A < \infty$.

(1) If $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{amp}X \geq \mathrm{amp}A$ and $n \geq 0$, TFAE:
(i) $\text{G-dim}_A X \leq n - \mathrm{sup}X$;
(ii) A has a sppj resolution $P_\bullet$ of X such that $X_n \in \mathcal{G}_0[-\mathrm{sup}X_n]$;
(iii) $X \in \mathcal{P}[-\mathrm{sup}X] * \cdots * \mathcal{P}[-\mathrm{sup}X + n - 1] * \mathcal{G}_0[-\mathrm{sup}X + n]$;
(iv) $X \in \mathcal{G}_0[-\mathrm{sup}X] * \mathcal{P}[-\mathrm{sup}X + 1] * \cdots * \mathcal{P}[-\mathrm{sup}X + n]$.

(2) If $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{amp}X < \mathrm{amp}A$ and $n \geq 0$, TFAE:
(i) $\text{G-dim}_A X \leq n + \mathrm{inf}A - \mathrm{inf}X$;
(ii) $X \oplus X[\mathrm{amp}A - \mathrm{amp}X] \in \mathcal{P}[-\mathrm{sup}X] * \cdots * \mathcal{P}[-\mathrm{sup}X + n - 1] * \mathcal{G}_0[-\mathrm{sup}X + n]$.

(3) TFAE:
(i) A is local Gorenstein ($\mathrm{amp}A < \infty$ and $\mathrm{injdim}_A A < \infty$);
(ii) $\text{G-dim}_A \bar{k} < \infty$;
(iii) $\text{G-dim}_A X < \infty$ for any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.

Theorem (HYZ)

$(A, \bar{\mathfrak{m}}, \bar{k})$: comm. noetherian local DG-ring with $\mathrm{amp}A < \infty$.

(1) If $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{amp}X \geq \mathrm{amp}A$ and $n \geq 0$, TFAE:
(i) $\text{G-dim}_A X \leq n - \mathrm{sup}X$;
(ii) A has a sppj resolution $P_\bullet$ of X such that $X_n \in \mathcal{G}_0[-\mathrm{sup}X_n]$;
(iii) $X \in \mathcal{P}[-\mathrm{sup}X] * \cdots * \mathcal{P}[-\mathrm{sup}X + n - 1] * \mathcal{G}_0[-\mathrm{sup}X + n]$;
(iv) $X \in \mathcal{G}_0[-\mathrm{sup}X] * \mathcal{P}[-\mathrm{sup}X + 1] * \cdots * \mathcal{P}[-\mathrm{sup}X + n]$.

(2) If $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{amp}X < \mathrm{amp}A$ and $n \geq 0$, TFAE:
(i) $\text{G-dim}_A X \leq n + \mathrm{inf}A - \mathrm{inf}X$;
(ii) $X \oplus X[\mathrm{amp}A - \mathrm{amp}X] \in \mathcal{P}[-\mathrm{sup}X] * \cdots * \mathcal{P}[-\mathrm{sup}X + n - 1] * \mathcal{G}_0[-\mathrm{sup}X + n]$.

(3) TFAE:
(i) A is local Gorenstein ($\mathrm{amp}A < \infty$ and $\mathrm{injdim}_A A < \infty$);
(ii) $\text{G-dim}_A \bar{k} < \infty$;
(iii) $\text{G-dim}_A X < \infty$ for any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.

Theorem (HYZ)

$(A, \bar{\mathfrak{m}}, \bar{k})$: comm. noetherian local DG-ring with $\mathrm{amp}A < \infty$.

(1) If $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{amp}X \geq \mathrm{amp}A$ and $n \geq 0$, TFAE:
(i) $\text{G-dim}_A X \leq n - \mathrm{sup}X$;
(ii) A has a sppj resolution $P_\bullet$ of X such that $X_n \in \mathcal{G}_0[-\mathrm{sup}X_n]$;
(iii) $X \in \mathcal{P}[-\mathrm{sup}X] * \cdots * \mathcal{P}[-\mathrm{sup}X + n - 1] * \mathcal{G}_0[-\mathrm{sup}X + n]$;
(iv) $X \in \mathcal{G}_0[-\mathrm{sup}X] * \mathcal{P}[-\mathrm{sup}X + 1] * \cdots * \mathcal{P}[-\mathrm{sup}X + n]$.

(2) If $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{amp}X < \mathrm{amp}A$ and $n \geq 0$, TFAE:
(i) $\text{G-dim}_A X \leq n + \mathrm{inf}A - \mathrm{inf}X$;
(ii) $X \oplus X[\mathrm{amp}A - \mathrm{amp}X] \in \mathcal{P}[-\mathrm{sup}X] * \cdots * \mathcal{P}[-\mathrm{sup}X + n - 1] * \mathcal{G}_0[-\mathrm{sup}X + n]$.

(3) TFAE:
(i) A is local Gorenstein ($\mathrm{amp}A < \infty$ and $\mathrm{injdim}_A A < \infty$);
(ii) $\text{G-dim}_A \bar{k} < \infty$;
(iii) $\text{G-dim}_A X < \infty$ for any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.

Remark

1. The inequality $\mathrm{amp}X < \mathrm{amp}A$ is very often met. For instance, assume that $(A, \bar{\mathfrak{m}}, \bar{k})$ is a comm. noetherian local DG-ring with $0 < \mathrm{amp}A < \infty$. Set $X = \bar{k}$. Then $\mathrm{amp}X = 0 < \mathrm{amp}A$.
2. If $(A, \bar{\mathfrak{m}}, \bar{k})$ is a local Gorenstein DG-ring with $0 < \mathrm{amp}A < \infty$, then $\bar{k}$ has finite G-dimension by Theorem, but $\bar{k}$ never has finite projective dimension.
3. G-dmension is a finer invariant than projective dimension for DG-modules.

Remark

1. The inequality $\mathrm{amp}X < \mathrm{amp}A$ is very often met. For instance, assume that $(A, \bar{\mathfrak{m}}, \bar{k})$ is a comm. noetherian local DG-ring with $0 < \mathrm{amp}A < \infty$. Set $X = \bar{k}$. Then $\mathrm{amp}X = 0 < \mathrm{amp}A$.
2. If $(A, \bar{\mathfrak{m}}, \bar{k})$ is a local Gorenstein DG-ring with $0 < \mathrm{amp}A < \infty$, then $\bar{k}$ has finite G-dimension by Theorem, but $\bar{k}$ never has finite projective dimension.
3. G-dmension is a finer invariant than projective dimension for DG-modules.

Remark

1. The inequality $\mathrm{amp}X < \mathrm{amp}A$ is very often met. For instance, assume that $(A, \bar{\mathfrak{m}}, \bar{k})$ is a comm. noetherian local DG-ring with $0 < \mathrm{amp}A < \infty$. Set $X = \bar{k}$. Then $\mathrm{amp}X = 0 < \mathrm{amp}A$.
2. If $(A, \bar{\mathfrak{m}}, \bar{k})$ is a local Gorenstein DG-ring with $0 < \mathrm{amp}A < \infty$, then $\bar{k}$ has finite G-dimension by Theorem, but $\bar{k}$ never has finite projective dimension.
3. G-dmension is a finer invariant than projective dimension for DG-modules.

Remark

1. The inequality $\mathrm{amp}X < \mathrm{amp}A$ is very often met. For instance, assume that $(A, \bar{\mathfrak{m}}, \bar{k})$ is a comm. noetherian local DG-ring with $0 < \mathrm{amp}A < \infty$. Set $X = \bar{k}$. Then $\mathrm{amp}X = 0 < \mathrm{amp}A$.
2. If $(A, \bar{\mathfrak{m}}, \bar{k})$ is a local Gorenstein DG-ring with $0 < \mathrm{amp}A < \infty$, then $\bar{k}$ has finite G-dimension by Theorem, but $\bar{k}$ never has finite projective dimension.
3. G-dmension is a finer invariant than projective dimension for DG-modules.

Bass, Finitistic Dimension Conjecture

$\mathrm{fpd}A = \sup\{\mathrm{projdim}_A M \,|\, M \text{ f.g. with } \mathrm{projdim}_A M < \infty\} < \infty$ holds for a Artin algebra A.

Bird, Shaul, Sridhar and Williamson

Let A be a comm. noetherian DG-rings with bounded cohomology. The *little finitistic dimension* of A,

$$\mathrm{fpd}A = \sup\{\mathrm{projdim}_A M + \mathrm{inf} M \,|\, M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) \text{with } \mathrm{projdim}_A M < \infty\}.$$

[14] H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. 95 (1960) 466–488.

Bass, Finitistic Dimension Conjecture

$\mathrm{fpd}A = \sup\{\mathrm{projdim}_A M \mid M \text{ f.g. with } \mathrm{projdim}_A M < \infty\} < \infty$ holds for a Artin algebra A.

Bird, Shaul, Sridhar and Williamson

Let A be a comm. noetherian DG-rings with bounded cohomology. The *little finitistic dimension* of A,

$$\mathrm{fpd}A = \sup\{\mathrm{projdim}_A M + \inf M \mid M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) \text{with } \mathrm{projdim}_A M < \infty\}.$$

[14] H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. 95 (1960) 466–488.

Xi, 2008

The little finitistic dimensions of an algebra can alternatively be computed by G-dimension, that is,

$$\mathrm{fpd}A = \sup\{\text{G-dim}_A M \,|\, M \text{ f.g. with G-dim}_A M < \infty\}.$$

Corollary 1

If A is a comm. noetherian local DG-ring with $\mathrm{amp}A < \infty$, then

$$\mathrm{fpd}A = \sup\{\text{G-dim}_A X + \inf X \mid X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) \text{ with G-dim}_A X < \infty\}.$$

[15] C.C. Xi, *On the finitistic dimension conjecture, III: Related to the pair $eAe \subseteq A$*, J. Algebra 319 (2008) 3666–3688.

[16] I. Bird, L. Shaul, P. Sridhar and J. Williamson, *Finitistic dimensions over commutative DG-rings*, arXiv:2204.06865v2, 2022.

Xi, 2008

The little finitistic dimensions of an algebra can alternatively be computed by G-dimension, that is,

$$\mathrm{fpd}A = \sup\{\text{G-dim}_A M \,|\, M \text{ f.g. with G-dim}_A M < \infty\}.$$

Corollary 1

If A is a comm. noetherian local DG-ring with $\mathrm{amp}A < \infty$, then

$$\mathrm{fpd}A = \sup\{\text{G-dim}_A X + \inf X \mid X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) \text{ with G-dim}_A X < \infty\}.$$

[15] C.C. Xi, *On the finitistic dimension conjecture, III: Related to the pair* $eAe \subseteq A$, J. Algebra 319 (2008) 3666–3688.

[16] I. Bird, L. Shaul, P. Sridhar and J. Williamson, *Finitistic dimensions over commutative DG-rings*, arXiv:2204.06865v2, 2022.

Let A be a comm. noetherian local ring (not a DG-ring). A close relation between the class of maximal Cohen-Macaulay modules

$$\mathcal{M} = \{M \in \mathrm{Mod}A \,|\, \mathrm{depth}_A M = \dim A\}$$

and the class $\mathcal{G} = \mathcal{G}_0$ of modules of G-dimension zero over A can be shown in the following fact.

1. A is Cohen-Macaulay iff $\mathcal{G} \subseteq \mathcal{M}$.
2. A is Gorenstein iff $\mathcal{G} = \mathcal{M}$.

---

[17] L.W. Christensen, G. Piepmeyer, J. Striuli and R. Takahashi, *Finite Gorenstein representation type implies simple singularity*, Adv. Math. 218 (2008) 1012–1026.

Let A be a comm. noetherian local ring (not a DG-ring). A close relation between the class of maximal Cohen-Macaulay modules

$$\mathcal{M} = \{M \in \mathrm{Mod}A | \mathrm{depth}_A M = \mathrm{dim}A\}$$

and the class $\mathcal{G} = \mathcal{G}_0$ of modules of G-dimension zero over A can be shown in the following fact.

1. A is Cohen-Macaulay iff $\mathcal{G} \subseteq \mathcal{M}$.
2. A is Gorenstein iff $\mathcal{G} = \mathcal{M}$.

[17] L.W. Christensen, G. Piepmeyer, J. Striuli and R. Takahashi, *Finite Gorenstein representation type implies simple singularity*, Adv. Math. 218 (2008) 1012–1026.

Let A be a comm. noetherian local ring (not a DG-ring). A close relation between the class of maximal Cohen-Macaulay modules

$$\mathcal{M} = \{M \in \mathrm{Mod}A | \mathrm{depth}_A M = \dim A\}$$

and the class $\mathcal{G} = \mathcal{G}_0$ of modules of G-dimension zero over A can be shown in the following fact.

1. A is Cohen-Macaulay iff $\mathcal{G} \subseteq \mathcal{M}$.
2. A is Gorenstein iff $\mathcal{G} = \mathcal{M}$.

---

[17] L.W. Christensen, G. Piepmeyer, J. Striuli and R. Takahashi, *Finite Gorenstein representation type implies simple singularity*, Adv. Math. 218 (2008) 1012–1026.

Let A be a comm. noetherian local ring (not a DG-ring). A close relation between the class of maximal Cohen-Macaulay modules

$$\mathcal{M} = \{M \in \mathrm{Mod}A | \mathrm{depth}_A M = \dim A\}$$

and the class $\mathcal{G} = \mathcal{G}_0$ of modules of G-dimension zero over A can be shown in the following fact.

1. A is Cohen-Macaulay iff $\mathcal{G} \subseteq \mathcal{M}$.
2. A is Gorenstein iff $\mathcal{G} = \mathcal{M}$.

---

[17] L.W. Christensen, G. Piepmeyer, J. Striuli and R. Takahashi, *Finite Gorenstein representation type implies simple singularity*, Adv. Math. 218 (2008) 1012–1026.

For a comm. noetherian DG-ring A and $X \in \mathrm{D}^{-}(A)$, denote

$$\mathrm{lc.dim}_A X := \sup_{l \in \mathbb{Z}} \{\dim(\mathrm{H}^l(M)) + l\}.$$

We can generalize the above fact to the setting of comm. noetherian local DG-rings.

Corollary 2

Let $(A, \bar{\mathfrak{m}})$ be a comm. noetherian local DG-ring with $\mathrm{amp}A < \infty$. Set $\mathcal{H} = \{X \in \mathcal{G} \mid \mathrm{lc.dim}_A X - \mathrm{depth}_A X \geq \mathrm{amp}X = \mathrm{amp}A\}$. Then

1. A is local CM iff $\mathcal{H} \subseteq \mathcal{M} = \{X \in \mathcal{G} \mid \mathrm{lc.dim}_A X - \mathrm{depth}_A X = \mathrm{amp}X = \mathrm{amp}A, \mathrm{lc.dim}_A X - \sup X = \dim \mathrm{H}^0(A)\}$ the class of maximal Cohen-Macaulay DG-modules.
2. A is Gorentein iff $\mathcal{H} = \mathcal{M}$.

For a comm. noetherian DG-ring A and $X \in \mathrm{D}^-(A)$, denote

$$\mathrm{lc.dim}_A X := \sup_{l \in \mathbb{Z}} \{\dim(\mathrm{H}^l(M)) + l\}.$$

We can generalize the above fact to the setting of comm. noetherian local DG-rings.

Corollary 2

Let $(A, \bar{\mathfrak{m}})$ be a comm. noetherian local DG-ring with $\mathrm{amp}A < \infty$. Set $\mathcal{H} = \{X \in \mathcal{G} \mid \mathrm{lc.dim}_A X - \mathrm{depth}_A X \geq \mathrm{amp}X = \mathrm{amp}A\}$. Then

1. A is local CM iff $\mathcal{H} \subseteq \mathcal{M} = \{X \in \mathcal{G} \mid \mathrm{lc.dim}_A X - \mathrm{depth}_A X = \mathrm{amp}X = \mathrm{amp}A, \mathrm{lc.dim}_A X - \sup X = \dim \mathrm{H}^0(A)\}$ the class of maximal Cohen-Macaulay DG-modules.
2. A is Gorentein iff $\mathcal{H} = \mathcal{M}$.

For a comm. noetherian DG-ring A and $X \in \mathrm{D}^{-}(A)$, denote

$$\mathrm{lc.dim}_A X := \sup_{l \in \mathbb{Z}} \{\dim(\mathrm{H}^l(M)) + l\}.$$

We can generalize the above fact to the setting of comm. noetherian local DG-rings.

Corollary 2

Let $(A, \bar{\mathfrak{m}})$ be a comm. noetherian local DG-ring with $\mathrm{amp} A < \infty$. Set $\mathcal{H} = \{X \in \mathcal{G} \mid \mathrm{lc.dim}_A X - \mathrm{depth}_A X \geq \mathrm{amp} X = \mathrm{amp} A\}$. Then

1. A is local CM iff $\mathcal{H} \subseteq \mathcal{M} = \{X \in \mathcal{G} \mid \mathrm{lc.dim}_A X - \mathrm{depth}_A X = \mathrm{amp} X = \mathrm{amp} A, \mathrm{lc.dim}_A X - \sup X = \dim \mathrm{H}^0(A)\}$ the class of maximal Cohen-Macaulay DG-modules.
2. A is Gorentein iff $\mathcal{H} = \mathcal{M}$.

Buchweitz introduced the *singularity category*,

$$\mathrm{D_{sg}}(A) := \mathrm{D^b}(A)/\mathrm{per}(A).$$

It measures the homological singularity of the category of f.g. left A-modules in sense that $\mathrm{D_{sg}}(A) = 0$ iff $\mathrm{gldim} A < \infty$.

Buchweitz-Happel Theorem

$\exists$ a fully faithful triangle functor $F : \underline{\mathcal{M}} \to \mathrm{D}_{sg}(A)$ provided that A is a comm. local Gorenstein ring.

Bergh, Jørgensen, Oppermann

$\exists$ a fully faithful triangle functor $F : \underline{\mathcal{M}} \to \mathrm{D}_{sg}(A)$ iff A is a comm. local Gorenstein ring.

[19] P.A. Bergh, D.A. Jørgensen, S. Oppermann, *The Gorenstein defect category*, Quart. J. Math. 66(2) (2015) 459–471.

[20] D. Happel, *On Gorenstein Algebras*, in: Representation theory of finite groups and finite-dimensional algebras (Proc. Conf. at Bielefeld, 1991), Progress in Math. 95, Birkhäuser, Basel, 1991, pp.389–404.

Buchweitz introduced the *singularity category*,

$$\mathrm{D_{sg}}(A) := \mathrm{D^b}(A)/\mathrm{per}(A).$$

It measures the homological singularity of the category of f.g. left A-modules in sense that $\mathrm{D_{sg}}(A) = 0$ iff $\mathrm{gldim} A < \infty$.

Buchweitz-Happel Theorem

$\exists$ a fully faithful triangle functor $F : \underline{\mathcal{M}} \to \mathrm{D}_{sg}(A)$ provided that A is a comm. local Gorenstein ring.

Bergh, Jørgensen, Oppermann

$\exists$ a fully faithful triangle functor $F : \underline{\mathcal{M}} \to \mathrm{D}_{sg}(A)$ iff A is a comm. local Gorenstein ring.

[19] P.A. Bergh, D.A. Jørgensen, S. Oppermann, *The Gorenstein defect category*, Quart. J. Math. 66(2) (2015) 459–471.

[20] D. Happel, *On Gorenstein Algebras*, in: Representation theory of finite groups and finite-dimensional algebras (Proc. Conf. at Bielefeld, 1991), Progress in Math. 95, Birkhäuser, Basel, 1991, pp.389–404.

Buchweitz-Happel Theorem has been generalized by Jin to a proper noncommutative Gorenstein DG-algebra over a field k.

Jin, [21, Theorem 0.3 and Assumption 0.1]

A: Gorenstein DG-algebra over a field $\Rightarrow \exists$ triangle equivalence

$$F : \underline{\mathrm{CM}}(A) \to \mathrm{D}_{sg}(A) := \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)/\langle \mathcal{P} \rangle.$$

Corollary 3

A: comm. noetherian local DG-ring with $\mathrm{amp} A < \infty$. Let $\mathcal{A} = \{X \in \mathcal{R}(A) \mid \sup X \le 0 \ \text{ and } \ \text{G-dim}_A X \le 0\}$. Then the functor

$$F : \underline{\mathcal{A}} \to \mathrm{D}_{sg}(A)$$

is a triangle equivalence iff A is a local Gorenstein DG-ring.

[21] H.B. Jin, *Cohen-Macaulay differential graded modules and negative Calabi-Yau configurations*, Adv. Math. 374 (2020) 107338.

Buchweitz-Happel Theorem has been generalized by Jin to a proper noncommutative Gorenstein DG-algebra over a field k.

Jin, [21, Theorem 0.3 and Assumption 0.1]

A: Gorenstein DG-algebra over a field $\Rightarrow \exists$ triangle equivalence

$$F : \underline{\mathrm{CM}}(A) \to \mathrm{D}_{sg}(A) := \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)/\langle \mathcal{P} \rangle.$$

Corollary 3

A: comm. noetherian local DG-ring with $\mathrm{amp} A < \infty$. Let $\mathcal{A} = \{X \in \mathcal{R}(A) \mid \sup X \leq 0 \text{ and } \mathrm{G}\text{-}\mathrm{dim}_A X \leq 0\}$. Then the functor

$$F : \underline{\mathcal{A}} \to \mathrm{D}_{sg}(A)$$

is a triangle equivalence iff A is a local Gorenstein DG-ring.

---

[21] H.B. Jin, *Cohen-Macaulay differential graded modules and negative Calabi-Yau configurations*, Adv. Math. 374 (2020) 107338.

Thank you!