A recollement approach to Han's conjecture

Xiaoxiao Xu(East China Normal University)

Shanghai Jiao Tong University
August 2024

Joint work

This talk is based on joint work

- Ren Wang(汪任), Xiaoxiao Xu (徐校校), Jinbi Zhang (张金币)and Guodong Zhou (周国栋), A recollement approach to Han's conjecture, preprint in preparation.

Plan

- Han's conjecture
- A reduction theorem and its proof
- Morita context algebras and Han's conjecture
- Applications

Part I: Happel's question

Let k be a field of arbitrary characteristic. Let A be a k-algebra.

Definition

Hochschild cohomology $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A \otimes A^{op}}(A, A)$, $n \geq 0$.

Theorem (Happel 1987)

Let k be an algebraically closed field, A be a finite dimensional k-algebra. If $\mathrm{gldim}(A) < \infty$, *then* $\mathrm{HH}^n(A) = 0, \forall n >> 0$.

In 1987, D. Happel asked the following question:

Question (Happel 1987)

Let A be a finite-dimensional algebra over a field k. If $\mathrm{HH}^n(A) = 0, \forall n >> 0$, *then is* $\mathrm{gldim}(A) < \infty$?

D. Happel, *Hochschild cohomology of finite-dimensional algebras*, in: Springer Lecture Notes in Mathematics, vol. 1404, 1989, pp. 108 – 126.

Part I: Happel's question

Let k be a field of arbitrary characteristic. Let A be a k-algebra.

Definition

Hochschild cohomology $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A \otimes A^{op}}(A, A)$, $n \geq 0$.

Theorem (Happel 1987)

Let k be an algebraically closed field, A be a finite dimensional k-algebra. If $\mathrm{gldim}(A) < \infty$, *then* $\mathrm{HH}^n(A) = 0, \forall n >> 0$.

In 1987, D. Happel asked the following question:

Question (Happel 1987)

Let A be a finite-dimensional algebra over a field k. If $\mathrm{HH}^n(A) = 0, \forall n >> 0$, *then is* $\mathrm{gldim}(A) < \infty$*?*

D. Happel, *Hochschild cohomology of finite-dimensional algebras*, in: Springer Lecture Notes in Mathematics, vol. 1404, 1989, pp. 108 – 126.

Part I: The answer to Happel's question is negative

Example (Buchweitz-Green-Madsen-Solberg 2005)

Let k be a field and $q \in k$. Let $A = k\langle x, y\rangle/(x^2, y^2, xy + qyx)$ with q not a root of unity. Then

$$\dim_k \mathrm{HH}^i(A) = \begin{cases} 2 & i = 0, 1, \\ 1 & i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

R.-O. Buchweitz, E.L. Green, D. Madsen, Ø. Solberg, *Finite Hochschild cohomology without finite global dimension*, Math. Res. Lett. **12** (2005) 805 – 816.

Schulz, R., *A nonprojective module without self-extensions*, Arch. Math. (Basel) **62** (1994) 497 – 500.

Part I: Han's conjecture

Definition

Hochschild homology $\mathrm{HH}_n(A) = \mathrm{Tor}_n^{A \otimes A^{op}}(A, A), n \geq 0.$

Yang Han (韩阳) asked the following question:

Conjecture (Han 2006)

Let A be a finite-dimensional algebra over a field k. if $\mathrm{HH}_n(A) = 0, \forall n >> 0$ *, then* $\mathrm{gldim}(A) < \infty$.

Theorem (Han 2006, Keller 98)

Assume that k is a perfect field, A is a finite dimensional algebra. If $\mathrm{gldim}(A) < \infty$*, then* $\mathrm{HH}_n(A) = 0, \forall n > 0.$

Yang Han (韩阳), *Hochschild (co)homology dimension.* J. London Math. Soc. (2) 73 (2006), no. 3, 657 – 668.

B. Keller, *Invariance and Localization for Cyclic Homology of DG algebras*, J. Pure Appl. Algebra **123** (1998), 223-273.

Part I: Han's conjecture

Definition

Hochschild homology $\mathrm{HH}_n(A) = \mathrm{Tor}_n^{A\otimes A^{op}}(A, A), n \geq 0.$

Yang Han (韩阳) asked the following question:

Conjecture (Han 2006)

Let A be a finite-dimensional algebra over a field k. if $\mathrm{HH}_n(A) = 0, \forall n >> 0$ *, then* $\mathrm{gldim}(A) < \infty$ *.*

Theorem (Han 2006, Keller 98)

Assume that k is a perfect field, A is a finite dimensional algebra. If $\mathrm{gldim}(A) < \infty$*, then* $\mathrm{HH}_n(A) = 0, \forall n > 0.$

- Yang Han (韩阳), *Hochschild (co)homology dimension*. J. London Math. Soc. (2) 73 (2006), no. 3, 657 – 668.
- B. Keller, *Invariance and Localization for Cyclic Homology of DG algebras*, J. Pure Appl. Algebra **123** (1998), 223-273.

Part I: Known cases for Han's conjecture

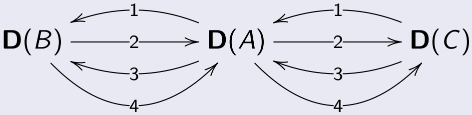
Han's conjecture is known to be true for

- monomial algebras, and truncated quiver algebras,
- commutative algebras,
- generalised Weyl algebras
- graded local algebras, Koszul algebras and graded cellular algebras under the condition that the ground field has characteristic 0
- etc

Part II: Our first result: A reduction theorem for Han's conjecture

Theorem (X.-Wang-Zhang-Zhou.2024)

Given a ladder of height two of unbounded derived categories of finite dimensional algebras



then Han's conjecture holds for A iff it holds for B and C.
So Han's conjecture is reduced to derived 2-simple algebras.

Ren Wang (汪任), Xiaoxiao Xu (徐校校), Jinbi Zhang (张金币)and Guodong Zhou (周国栋), A recollement approach to Han's conjecture, preprint in preparation.

Part II: Notations

Let A be a finite dimensional algebra defined over a field k.

Notation

- $\mathrm{Mod}\ A$ the category of all right A-modules,
- $\mathrm{mod}\ A$ the category of finitely generated right A-modules,
- $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}\ A)$ the bounded derived category of $\mathrm{mod}\ A$
- $\mathbf{D}(A) = \mathbf{D}(\mathrm{Mod}\ A)$ the unbounded derived category of $\mathrm{Mod}\ A$

Part II: Short exact sequences

A sequence of triangle functors

$$\mathcal{T}' \xrightarrow{i_*} \mathcal{T} \xrightarrow{j^*} \mathcal{T}''$$

is called a **short exact sequence up to direct summands** if i_* is fully faithful, $j^* \circ i_* = 0$ and the induced functor $\overline{j^*} : \mathcal{T}/\mathcal{T}' \to \mathcal{T}''$ is also fully faithful, moreover, $\overline{j^*}$ is dense up to direct summands (i.e. for each object Y of $\mathcal{T}''$, there exists $X \in \mathcal{T}$ such that Y is a direct summand of $\overline{j^*}(X)$).

A short exact sequence up to direct summands is a **short exact sequence** if furthermore, $\mathcal{T}'$ is a thick subcategory of $\mathcal{T}$ and the induced functor $\overline{j^*} : \mathcal{T}/\mathcal{T}' \to \mathcal{T}''$ is an equivalence.

Part II: Left/right recollements

Given a short exact sequence of triangulated categories

$$\mathcal{T}' \xrightarrow{i_*} \mathcal{T} \xrightarrow{j^*} \mathcal{T}/\mathcal{T}',$$

if i_* and/or j^* have a left adjoint, denoted by i^* and $j_!$ respectively, then the diagram

$$\mathcal{T}' \underset{\overset{i^*}{\longleftarrow}}{\xrightarrow{\;i_*\;}} \mathcal{T} \underset{\overset{j_!}{\longleftarrow}}{\xrightarrow{\;j^*\;}} \mathcal{T}''$$

is called a **colocalisation sequence** (or a **left recollement**) of triangulated categories.

Dually, given a short exact sequence of triangulated categories

$$\mathcal{T}' \xrightarrow{i_*} \mathcal{T} \xrightarrow{j^*} \mathcal{T}/\mathcal{T}',$$

if i_* and/or j^* have a right adjoint, denoted by $i^!$ and j_* respectively, then the diagram

$$\mathcal{T}' \underset{\overset{i^!}{\longleftarrow}}{\xrightarrow{\;i_*\;}} \mathcal{T} \underset{\overset{j_*}{\longleftarrow}}{\xrightarrow{\;j^*\;}} \mathcal{T}''$$

is called a **localisation sequence** (or a **right recollement**) of triangulated categories.

Part II: Left/right recollements

Given a short exact sequence of triangulated categories

$$\mathcal{T}' \xrightarrow{i_*} \mathcal{T} \xrightarrow{j^*} \mathcal{T}/\mathcal{T}',$$

if i_* and/or j^* have a left adjoint, denoted by i^* and $j_!$ respectively, then the diagram

$$\mathcal{T}' \underset{\xrightarrow{\quad i_* \quad}}{\xleftarrow{\quad i^* \quad}} \mathcal{T} \underset{\xrightarrow{\quad j^* \quad}}{\xleftarrow{\quad j_! \quad}} \mathcal{T}''$$

is called a **colocalisation sequence** (or a **left recollement**) of triangulated categories.

Dually, given a short exact sequence of triangulated categories

$$\mathcal{T}' \xrightarrow{i_*} \mathcal{T} \xrightarrow{j^*} \mathcal{T}/\mathcal{T}',$$

if i_* and/or j^* have a right adjoint, denoted by $i^!$ and j_* respectively, then the diagram

$$\mathcal{T}' \overset{\xrightarrow{\quad i_* \quad}}{\xleftarrow[\quad i^! \quad]{}} \mathcal{T} \overset{\xrightarrow{\quad j^* \quad}}{\xleftarrow[\quad j_* \quad]{}} \mathcal{T}''$$

is called a **localisation sequence** (or a **right recollement**) of triangulated categories.

Part II: Recollements

When i_* and/or j^* have a left adjoint and a right adjoint, the diagram

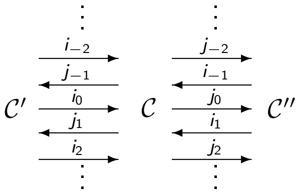
$$\xymatrix{\mathcal{T}' \ar[r]|{i_*} & \mathcal{T} \ar@/_1.5pc/[l]_{i^*} \ar@/^1.5pc/[l]^{i^!} \ar[r]|{j^*} & \mathcal{T}'' \ar@/_1.5pc/[l]_{j_!} \ar@/^1.5pc/[l]^{j_*}}$$

is called a **recollement** of triangulated categories. More precisely, it's called a recollement of $\mathcal{T}$ relative to $\mathcal{T}'$ and $\mathcal{T}''$.

A. A. Beilinson, J. N. Bernstein and P. Deligne, *Faisceaux pervers*, Astérisque **100**, Soc. Math. France, Paris, 1982.

Part II: Ladders

A **ladder** is a finite or an infinite diagram of triangle functors:



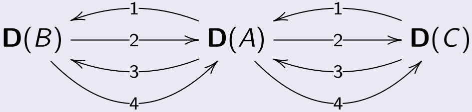
such that any two consecutive rows form a left or right recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$.

Its height is the number of rows minus 2.

Part II: Derived 2-simple algebras

Definition

A finite dimensional algebra A is called a derived 2-simple algebra if it DOES not admit any ladder of height two



with B, C finite dimensional algebras.

Remark

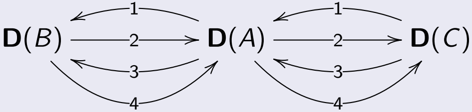
A finite dimensional algebra A is derived 2-simple iff it DOES not admit any right recollement

with B, C finite dimensional algebras.

Part II: Derived 2-simple algebras

Definition

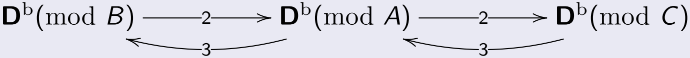
A finite dimensional algebra A is called a derived 2-simple algebra if it DOES not admit any ladder of height two



with B, C finite dimensional algebras.

Remark

A finite dimensional algebra A is derived 2-simple iff it DOES not admit any right recollement

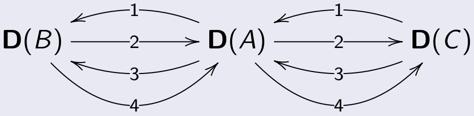


with B, C finite dimensional algebras.

Part II: A reduction theorem for Han's conjecture

Theorem (X.-Wang-Zhang-Zhou.2024)

Given a ladder of height two of unbounded derived categories of finite dimensional algebras



then Han's conjecture holds for A iff it holds for B and C.
So Han's conjecture is reduced to derived 2-simple algebras.

Ren Wang (汪任), Xiaoxiao Xu (徐校校), Jinbi Zhang (张金币)and Guodong Zhou (周国栋), A recollement approach to Han's conjecture, preprint in preparation.

Part II: Two ingredients

Theorem (Angeleri Hügel-Koenig-Liu-Yang 2017)

Assume that we are given a recollement of unbounded derived categories of finite dimensional algebras

$$\mathbf{D}(B) \;\underset{\overset{\longleftarrow\, i^! \,}{}}{\overset{\overset{\longleftarrow\, i^* \,}{}}{\xrightarrow{\; i_* = i_! \;}}}\; \mathbf{D}(A) \;\underset{\overset{\longleftarrow\, j_* \,}{}}{\overset{\overset{\longleftarrow\, j_! \,}{}}{\xrightarrow{\; j^* = j^! \;}}}\; \mathbf{D}(C)\,.$$

Then $\quad \mathrm{gldim}(A) < \infty \Leftrightarrow \mathrm{gldim}(B) < \infty \text{ and } \mathrm{gldim}(C) < \infty.$

Theorem (Keller, private communication)

Given a ladder of height two of unbounded derived categories of rings

$$\mathbf{D}(S) \;\begin{array}{c} \xleftarrow{\quad 1 \quad} \\ \xrightarrow{\quad 2 \quad} \\ \xleftarrow{\quad 3 \quad} \\ \xrightarrow{\quad 4 \quad} \end{array}\; \mathbf{D}(R) \;\begin{array}{c} \xleftarrow{\quad 1 \quad} \\ \xrightarrow{\quad 2 \quad} \\ \xleftarrow{\quad 3 \quad} \\ \xrightarrow{\quad 4 \quad} \end{array}\; \mathbf{D}(T)\,,$$

then

$$HH_n(R) \cong HH_n(S) \oplus HH_n(T), \forall n \in \mathbb{N}.$$

Part II: Two ingredients

Theorem (Angeleri Hügel-Koenig-Liu-Yang 2017)

Assume that we are given a recollement of unbounded derived categories of finite dimensional algebras

$$\mathbf{D}(B) \begin{matrix} \xleftarrow{\quad i^* \quad} \\ \xrightarrow{\quad i_*=i_! \quad} \\ \xleftarrow{\quad i^! \quad} \end{matrix} \mathbf{D}(A) \begin{matrix} \xleftarrow{\quad j_! \quad} \\ \xrightarrow{\quad j^*=j^! \quad} \\ \xleftarrow{\quad j_* \quad} \end{matrix} \mathbf{D}(C)\ .$$

Then $\quad \mathrm{gldim}(A) < \infty \Leftrightarrow \mathrm{gldim}(B) < \infty \text{ and } \mathrm{gldim}(C) < \infty.$

Theorem (Keller, private communication)

Given a ladder of height two of unbounded derived categories of rings

$$\mathbf{D}(S) \begin{matrix} \xleftarrow{\quad 1 \quad} \\ \xrightarrow{\quad 2 \quad} \\ \xleftarrow{\quad 3 \quad} \\ \xrightarrow{\quad 4 \quad} \end{matrix} \mathbf{D}(R) \begin{matrix} \xleftarrow{\quad 1 \quad} \\ \xrightarrow{\quad 2 \quad} \\ \xleftarrow{\quad 3 \quad} \\ \xrightarrow{\quad 4 \quad} \end{matrix} \mathbf{D}(T)\ ,$$

then

$$HH_n(R) \cong HH_n(S) \oplus HH_n(T), \forall n \in \mathbb{N}.$$

Part II: They go together

Theorem (X.-Wang-Zhang-Zhou. 2024)

Given a ladder of height two of derived categories of rings

$$\mathbf{D}(B) \begin{array}{c} \xleftarrow{\;1\;} \\ \xrightarrow{\;2\;} \\ \xleftarrow{\;3\;} \\ \xrightarrow{\;4\;} \end{array} \mathbf{D}(A) \begin{array}{c} \xleftarrow{\;1\;} \\ \xrightarrow{\;2\;} \\ \xleftarrow{\;3\;} \\ \xrightarrow{\;4\;} \end{array} \mathbf{D}(C)\ ,$$

then Han's conjecture holds for A iff it holds for B and C.

Proof.

By Angeleri Hügel-Koenig-Liu-Yang 2017,

$$\mathrm{gldim}(A) < \infty \Leftrightarrow \mathrm{gldim}(B) < \infty \text{ and } \mathrm{gldim}(C) < \infty.$$

By Keller,

$$\sum_{i=0}^{\infty} \dim_k \mathrm{HH}_i(A) < \infty \Longleftrightarrow \sum_{i=0}^{\infty} \dim_k \mathrm{HH}_i(B) < \infty \text{ and } \sum_{i=0}^{\infty} \dim_k \mathrm{HH}_i(C) < \infty$$

□

Part II: Localisation theorem for Hochschild homology

Theorem (Keller 1998)

Given a recollement of unbounded derived categories of rings

$$\mathbf{D}(S) \underset{\xleftarrow{3}}{\overset{\xleftarrow{1}}{\xrightarrow{2}}} \mathbf{D}(R) \underset{\xleftarrow{3}}{\overset{\xleftarrow{1}}{\xrightarrow{2}}} \mathbf{D}(T)\ ,$$

then there exists two long exact sequences

$$\cdots \to HH_n(T) \to HH_n(R) \to HH_n(S) \to HH_{n-1}(T) \to \cdots$$

$$\cdots \to HH_0(T) \to HH_0(R) \to HH_0(S) \to 0;$$

$$\cdots \to HC_n(T) \to HC_n(R) \to HC_n(S) \to HC_{n-1}(T) \to \cdots$$

$$\cdots \to HC_0(T) \to HC_0(R) \to HC_0(S) \to 0.$$

Part II: A splitting theorem for Hochschild homology and cyclic homology

Theorem (Keller, private communication)

Given a ladder of height two of derived categories of rings

$$\mathbf{D}(S) \;\begin{array}{c} \xleftarrow{\;1\;} \\ \xrightarrow{\;2\;} \\ \xleftarrow{\;3\;} \\ \xrightarrow{\;4\;} \end{array}\; \mathbf{D}(R) \;\begin{array}{c} \xleftarrow{\;1\;} \\ \xrightarrow{\;2\;} \\ \xleftarrow{\;3\;} \\ \xrightarrow{\;4\;} \end{array}\; \mathbf{D}(T)\;,$$

then

$$HH_n(R) \cong HH_n(S) \oplus HH_n(T), \forall n \in \mathbb{N}$$

and

$$HC_n(R) \cong HC_n(S) \oplus HC_n(T), \forall n \in \mathbb{N}.$$

B. Keller, *Invariance and Localization for Cyclic Homology of DG algebras*, J. Pure Appl. Algebra **123** (1998), 223-273.

B. Keller, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra **136** (1999), 1-56.

Part III: Stratifying ideals

Definition

Let $e = e^2 \in A$ be an idempotent in an algebra A. Denote $f = 1 - e$. Then the idempotent e (resp. the ideal AeA) is a stratifying idempotent (resp. ideal) if

- $\mathrm{Tor}_n^{eAe}(Ae, eA) = 0, n > 0;$
- *the natural map $fAe \otimes_{eAe} eAf \to fAf$ is injective.*

Theorem (Geigle-Lenzing 1991)

Let $e = e^2 \in A$ be a stratifying idempotent in an algebra A. Then there exists a recollement

$$\mathbf{D}(A/AeA) \underset{\xleftarrow{3}}{\overset{\xleftarrow{1}}{\xrightarrow{2}}} \mathbf{D}(A) \underset{\xleftarrow{3}}{\overset{\xleftarrow{1}}{\xrightarrow{2}}} \mathbf{D}(eAe)\ .$$

W. Geigle and H. Lenzing, *Perpendicular Categories with Applications to Representations and Sheaves*, J. Algebra **14** (1991), 273-343.

Part III: Stratifying ideals

Definition

Let $e = e^2 \in A$ be an idempotent in an algebra A. Denote $f = 1 - e$. Then the idempotent e (resp. the ideal AeA) is a stratifying idempotent (resp. ideal) if

- $\mathrm{Tor}_n^{eAe}(Ae, eA) = 0, n > 0;$
- *the natural map $fAe \otimes_{eAe} eAf \to fAf$ is injective.*

Theorem (Geigle-Lenzing 1991)

Let $e = e^2 \in A$ be a stratifying idempotent in an algebra A. Then there exists a recollement

$$\mathbf{D}(A/AeA) \;\underset{\xleftarrow{\;3\;}}{\overset{\xleftarrow{\;1\;}}{\xrightarrow{\;2\;}}}\; \mathbf{D}(A) \;\underset{\xleftarrow{\;3\;}}{\overset{\xleftarrow{\;1\;}}{\xrightarrow{\;2\;}}}\; \mathbf{D}(eAe)\ .$$

W. Geigle and H. Lenzing, *Perpendicular Categories with Applications to Representations and Sheaves*, J. Algebra **14** (1991), 273-343.

Part III: Triangular matrix algebras as examples

Let

$$A = \begin{pmatrix} B & 0 \\ {}_C M_B & C \end{pmatrix}$$

be a finite dimensional lower triangular matrix algebra.

Theorem (Cline-Parshall-Scott 1996)

Both idempotents $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *are stratifying and there exists a ladder of height two*

$$\mathbf{D}(B) \underset{\xrightarrow{\quad 4 \quad}}{\overset{\xleftarrow{\quad 1 \quad}}{\underset{\xleftarrow{\quad 3 \quad}}{\xrightarrow{\quad 2 \quad}}}} \mathbf{D}(A) \underset{\xrightarrow{\quad 4 \quad}}{\overset{\xleftarrow{\quad 1 \quad}}{\underset{\xleftarrow{\quad 3 \quad}}{\xrightarrow{\quad 2 \quad}}}} \mathbf{D}(C)\ .$$

E. Cline, B. Parshall, L. Scott, *Stratifying endomorphism algebras*, Mem. Amer. Math. Soc. **591** (1996) 1-119.

Part III: Han's conjecture for triangular algebras

Theorem (Cibils-Lanzilotta-Marcos-Solotar 2021; X.-Wang-Zhang-Zhou.2024)

Let

$$A = \begin{pmatrix} B & 0 \\ {}_C M_B & C \end{pmatrix}$$

be a finite dimensional lower triangular matrix algebra. Han's conjecture holds for A iff it holds for B and C.

C. Cibils, E. N. Marcos, A. Solotar, *Han's conjecture and Hochschild homology for null-square projective algebras*. Indiana Univ. Math. J. **70** (2021), no. 2, 639-668.

Ren Wang (汪任), Xiaoxiao Xu (徐校校), Jinbi Zhang (张金币)and Guodong Zhou (周国栋), A recollement approach to Han's conjecture, preprint in preparation.

Part III: Morita context algebras

Definition

A Morita context algebra is an algebra of the form $\begin{pmatrix} B & {}_BN_C \\ {}_CM_B & C \end{pmatrix}_{(\alpha,\beta)}$

where

- B, C *are algebras,* ${}_CM_B, {}_BN_C$ *are bimodules;*
- $\alpha : {}_BN \otimes_C M_B \to {}_BB_B$ *and* $\beta : {}_CM \otimes_B N_C \to {}_CC_C$ *are bimodule maps such that*

$$\alpha(n \otimes m)n' = n\beta(m \otimes n'), \beta(m \otimes n)m' = m\alpha(n \otimes m')$$

Fact

Morita context algebras are the same as algebras with an idempotent. More precisely, let $e = e^2 \in A$ *be an idempotent in an algebra* A*. Then* A *can be written as a Morita context algebra*

$$A = \begin{pmatrix} eAe & eAf \\ fAe & fAf \end{pmatrix}$$

Part III: Morita context algebras and stratifying idempotents

Proposition (Gao-Psaroudakis, 2017)

The idempotent $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *is stratifying iff*

- $\mathrm{Tor}_n^B(M, N) = 0, n > 0,$
- $\beta : M \otimes_B N \to C$ *is injective,*

and in this case, there is a recollement

$$\mathbf{D}(C/\mathrm{Im}(\beta)) \underset{\xleftarrow{3}}{\overset{\xleftarrow{1}}{\xrightarrow{2}}} \mathbf{D}(A) \underset{\xleftarrow{3}}{\overset{\xleftarrow{1}}{\xrightarrow{2}}} \mathbf{D}(B) .$$

Nan Gan (高楠) and C. Psaroudakis, *Gorenstein homological aspects of monomorphism categories via Morita rings*, Algebr. Represent. Theory **20** (2017), no. 2, 487-529.

Part III: Our second result

Theorem (X.-Wang-Zhang-Zhou. 2024)

Given a Morita context algebra $A = \begin{pmatrix} B & N \\ M & C \end{pmatrix}_{(\alpha,\beta)}$*, assume that* $\beta : M \otimes_B N \to C$ *is injective,* $\mathrm{Tor}_i^B(M, N) = 0, \forall i > 0$ *and that* M_B *has finite projective dimension. Then Han's conjecture holds for* A *iff it holds for* B *and* $C/\mathrm{Im}(\beta)$*.*

In fact, by Gao-Psaroudakis, 2017, under the hypothesis of the above theorem, there is a ladder of height two

$$\mathbf{D}(C/\mathrm{Im}(\beta)) \;\begin{smallmatrix} \xleftarrow{\;1\;} \\ \xrightarrow{\;2\;} \\ \xleftarrow{\;3\;} \\ \xrightarrow{\;4\;} \end{smallmatrix}\; \mathbf{D}(A) \;\begin{smallmatrix} \xleftarrow{\;1\;} \\ \xrightarrow{\;2\;} \\ \xleftarrow{\;3\;} \\ \xrightarrow{\;4\;} \end{smallmatrix}\; \mathbf{D}(B)\ .$$

Nan Gan (高楠) and C. Psaroudakis, *Gorenstein homological aspects of monomorphism categories via Morita rings*, Algebr. Represent. Theory **20** (2017), no. 2, 487-529.

Ren Wang (汪任), Xiaoxiao Xu (徐校校), Jinbi Zhang (张金币)and Guodong Zhou (周国栋), A recollement approach to Han's conjecture, preprint in preparation.

Part III: Our second result

Theorem (X.-Wang-Zhang-Zhou. 2024)

Given a Morita context algebra $A = \begin{pmatrix} B & {}_BN_C \\ {}_CM_B & C \end{pmatrix}_{(\alpha,\beta)}$, *assume that* $\alpha : N \otimes_C M \to B$ *is injective,* $\mathrm{Tor}_i^C(N, M) = 0, \forall i > 0$ *and that* N_C *has finite projective dimension. Then Han's conjecture holds for* A *iff it holds for* C *and* $B/\mathrm{Im}(\alpha)$.

In fact, by Gao-Psaroudakis, 2017, under the hypothesis of the above theorem, there is a ladder of height two

$$\mathbf{D}(B/\mathrm{Im}(\alpha)) \begin{array}{c} \xleftarrow{\quad 1 \quad} \\ \xrightarrow{\quad 2 \quad} \\ \xleftarrow{\quad 3 \quad} \\ \xrightarrow{\quad 4 \quad} \end{array} \mathbf{D}(A) \begin{array}{c} \xleftarrow{\quad 1 \quad} \\ \xrightarrow{\quad 2 \quad} \\ \xleftarrow{\quad 3 \quad} \\ \xrightarrow{\quad 4 \quad} \end{array} \mathbf{D}(C)\ .$$

- Nan Gan (高楠) and C. Psaroudakis, *Gorenstein homological aspects of monomorphism categories via Morita rings*, Algebr. Represent. Theory **20** (2017), no. 2, 487-529.
- Ren Wang (汪任), Xiaoxiao Xu (徐校校), Jinbi Zhang (张金币)and Guodong Zhou (周国栋), A recollement approach to Han's conjecture, preprint in preparation.

Part III: Examples

Example (Cibils-Marcos-Solotar 2021)

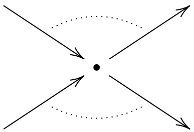
Given a Morita context algebra $A = \begin{pmatrix} B & {}_BN_C \\ {}_CM_B & C \end{pmatrix}_{(\alpha,\beta)}$, *assume that* M *and* N *are projective bimodules and* α, β *vanish. Then Han's conjecture holds for* A *iff it holds for* B *and* C.

C. Cibils, M. J. Redondo and A. Solotar, *Han's conjecture and Hochschild homology for null-square projective algebras*. Indiana Univ. Math. J. **70** (2021), no. 2, 639-668.

Part IV: Gentle algebras

Definition

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver and I be an admissible ideal. $A(Q, I)$ is called a gentle algebra if (Q, I) is a gentle pair i.e. it satisfies:
(1) Each vertex of Q is start point of at most two arrows, and end point of at most two arrows.
(2) For each arrow α in Q, there is at most one arrow β with $t(\alpha) = s(\beta)$ such that $\alpha\beta \notin I$, and at most one arrow γ with $t(\gamma) = s(\alpha)$ such that $\gamma\alpha \notin I$.
(3) For each arrow α in Q, there is at most one arrow β with $t(\alpha) = s(\beta)$ such that $\alpha\beta \in I$, and at most one arrow γ with $t(\gamma) = s(\alpha)$ such that $\gamma\alpha \in I$.
(4) The algebra $A(Q, I)$ is finite dimensional.



Part IV: Skew-gentle algebras

Definition

Let (Q, I) be a gentle pair. We add some special loops in Q, and denote the set of special loops by S_p. We call (Q, I, S_p) the skew-gentle triple if $(Q', I \cup \{\alpha^2 | \alpha \in S_p\})$ is a gentle pair where Q' is the quiver by adding the special loops to Q. The finite dimensional algebra $A(Q, I, S_p) = kQ'/\langle I \cup \{\alpha^2 - \alpha | \alpha \in S_p\}\rangle$ is called a skew-gentle algebra .

C. Geiss and J. A. De La Peña, *Auslander-Reiten components for clans*, Bol. Soc. Mat. Mexicana **5** (1999), no. 2, 307-326.

Theorem (X.-Wang-Zhang-Zhou. 2024)

Han's conjecture holds for skew-gentle algebras.

Proof.

Let $A = A(Q, I, S_p)$ be a skew-gentle algebra with $S_p \neq \emptyset$. Denote $B = A(Q, I)$ as the gentle algebra corresponding to A. By a result of Yiping Chen, there is a recollement of unbounded derived categories:

$$\mathbf{D}(B) \;\underset{i^!}{\overset{i^*}{\leftleftarrows}}\!\xrightarrow{\;i_*=i_!\;}\; \mathbf{D}(A) \;\underset{j_*}{\overset{j_!}{\leftleftarrows}}\!\xrightarrow{\;j^!=j^*\;}\; \mathbf{D}(C).$$

where C is a finite dimensional algebra with $gldim(C) \leq 1$. □

Y. P. Chen, *A Characteristic free approach to skew-gentle algebras.* arXiv:2212.06467.

Theorem (X.-Wang-Zhang-Zhou. 2024)

Han's conjecture holds for GLS algebras and finite EI category algebras.

- P. Webb, *An introduction to the representations and cohomology of categories, Group representation theory*, EPFL Press. Lausanne. (2007), 149-173.
- C. Geiss, B. Leclerc and J. Schroer, *Quivers with relations for symmetrizable Cartan matrices I: Foundations*, Invent. Math, **209** (2017), 61-158.

Thank you!