ICRA 21

Quasi-hereditary orderings of Nakayama algebras

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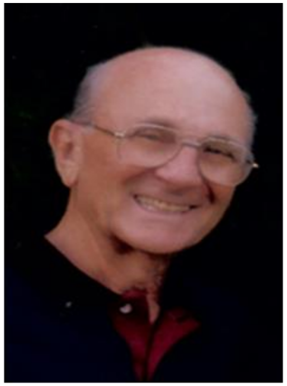
A joint work on arXiv:2405.02860 with Yuehui Zhang

9th, August, 2024

Plan

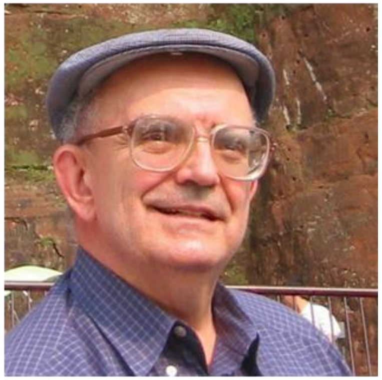
- Characterization of quasi-hereditary orderings
- Characterization of quasi-hereditary algebras
- Compute quasi-hereditary orderings

Quasi-hereditary Algebra



Edward Cline

Brian Parshall



Leonard Scott

The theory of quasi-hereditary algebras has been extensively studied since its introduction by E. Cline, B. Parshall, and L. Scott in their seminal paper [1] in 1988.

[1]E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.

Quasi-hereditary Orderings

The properties of a quasi-hereditary algebra A are heavily dependent on a specific ordering of the isomorphism classes of simple A -modules S.

This ordering ensures that every indecomposable projective module can be filtered by the Weyl modules, which are constructed through a special process related to the ordering. Such an ordering is termed a quasi-hereditary ordering on A.

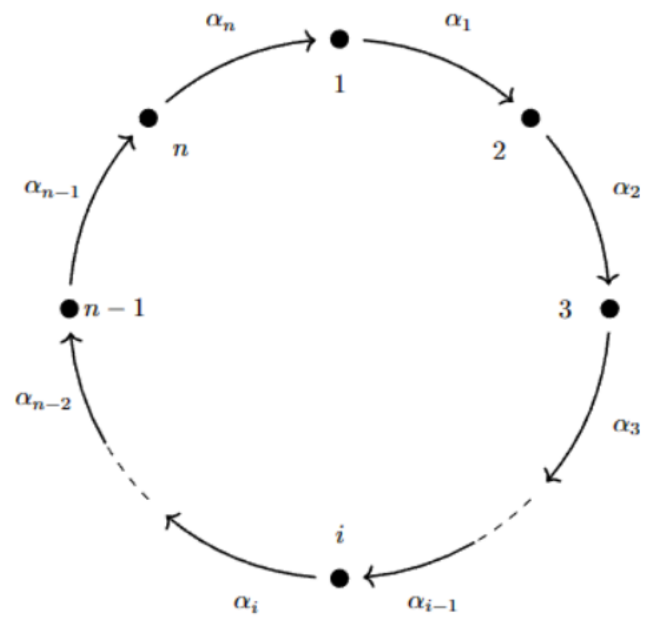
Nakayama Algebra

directed

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n$$

$$A_n$$

cyclic



$$\tilde{A}_n$$

Nakayama Algebra

Intro

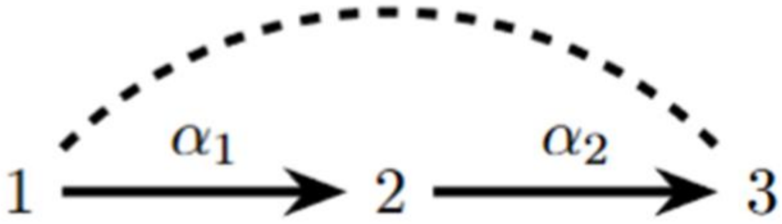
Let Q be any finite quiver, and let $A = KQ/I$ be a finite-dimensional algebra with I being an admissible ideal of KQ. For simplicity, we denote the simple A -module $S_x$corresponding to the vertex x (i.e., $S_x = A/AeA$ with $e = 1_A - x$) simply as x. Thus, the symbol x simultaneously represents a number, a vertex, and a simple module. Consequently, the weight poset of A can be chosen as the set S of all simple A -modules. This notation system prevents confusion when we use $\preccurlyeq$ (or its reverse order $\succcurlyeq$) to denote the partial order of the weight poset Λ.

Definition

Consider $g \in I$ to be a path of length(= the number of arrows +1) $l_g \geq 3$ in KQ.

The origin vertex of g is referred to as the "$hook$" of g, denoted as $h(g)$, while the end vertex of g is termed the "$denouement$" of g, denoted as $d(g)$.

$A = KA_3/< g >$
$g = \alpha_2\alpha_1$



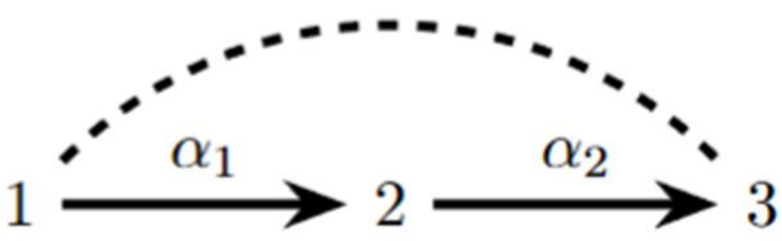
$h(g) = \{1\}$
$d(g) = \{3\}$

$hood, internal$

Definition

The ordered multiple set $\{h(g)\ , h(g) + 1\ ,\dots, d(g)\}$ of simple A-modules associated with g is called the "$hood$" of g and denoted as $Hod(g)$.
The multiple set $Hod(g)\backslash\{h(g), d(g)\}$ of simple A-modules is denoted by $Int(g)$, where the elements of $Int(g)$ are referred to as interior simple modules of g.
It is noteworthy that $|Hod(g)| = l_g$ and $|Int(g)| = l_g - 2$.

$A = KA_3/< g >$
$g = \alpha_2\alpha_1$



$Hod(g) = \{1,2,3\}$
$h(g) = \{1\}$
$d(g) = \{3\}$
$Int(g) = \{2\}$

Characterization of quasi-hereditary orderings

Theroem 1

Let $A = KQ_n/I$ be a Nakayama algebra with $I = < g_1, \dots, g_k >$. Let $\preccurlyeq$ be an ordering of A.
Then $\preccurlyeq$ is quasi-hereditary if and only if $maxHod(g_i) \notin Int(g_i)$, $\forall i = 1, \dots, k$.

$A = KA_3/< \alpha_2\alpha_1 >$

$1 \succcurlyeq 2 \succcurlyeq 3$ ✓

$1 \succcurlyeq 3 \succcurlyeq 2$ ✓

$3 \succcurlyeq 2 \succcurlyeq 1$ ✓

$3 \succcurlyeq 1 \succcurlyeq 2$ ✓

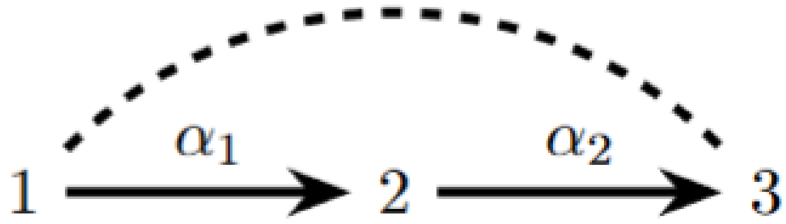
$2 \succcurlyeq 1 \succcurlyeq 3$ ×

$2 \succcurlyeq 3 \succcurlyeq 1$ ×

$q(A) = 4$

Characterization of quasi-hereditary orderings

$A = KA_3/< \alpha_2\alpha_1 >$



$1 \succcurlyeq 2 \succcurlyeq 3$ ✓

$1 \succcurlyeq 3 \succcurlyeq 2$ ✓

$3 \succcurlyeq 2 \succcurlyeq 1$ ✓

$3 \succcurlyeq 1 \succcurlyeq 2$ ✓

$2 \succcurlyeq 1 \succcurlyeq 3$ ×

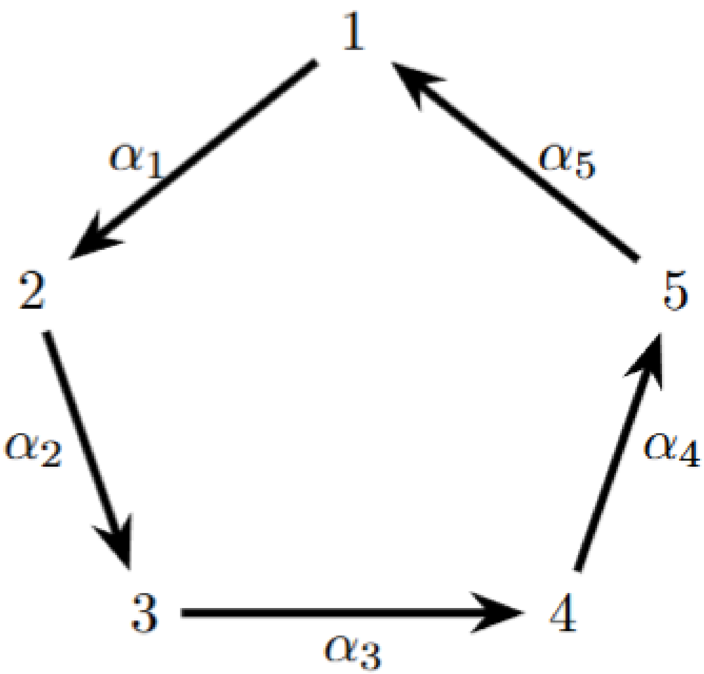
$2 \succcurlyeq 3 \succcurlyeq 1$ ×

$q(A) = 4$

Defn of $q(A)$

$q(A) :=$ ***the number of quasi − hereditary orderings of A***

An example



$$A = K\tilde{A}_5 / < \alpha_2\alpha_1, \alpha_3\alpha_2 >$$

2 $\neq$ max{1,2,3} and **3** $\neq$ max{2,3,4}

$\mathbf{2} \succcurlyeq 1 \succcurlyeq 3 \succcurlyeq 4 \succcurlyeq 5$ ✗

$1 \succcurlyeq \mathbf{3} \succcurlyeq 2 \succcurlyeq 4 \succcurlyeq 5$ ✗

$1 \succcurlyeq 2 \succcurlyeq 3 \succcurlyeq 4 \succcurlyeq 5$ ✓

$5 \succcurlyeq 4 \succcurlyeq 3 \succcurlyeq 2 \succcurlyeq 1$ ✓

$4 \succcurlyeq 3 \succcurlyeq 2 \succcurlyeq 5 \succcurlyeq 1$ ✓

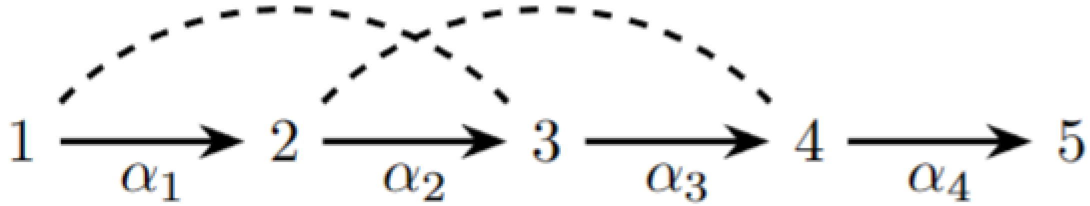
$$q(A) = \frac{1}{3} \times 5! = 40$$

Green-Schroll set $\mathcal{X}$

Defn of $\mathcal{X}$

For a Nakayama algebra $A = KQ_n/I$ with $I =< g_1, \ldots, g_k >$. The Green-Schroll set of A, denoted by $\mathcal{X}_A$ or simply $\mathcal{X}$, is the set $S - \bigcup_{i=1}^{k} Int(g_i)$.

$$\mathcal{X} = \{1,4,5\}$$



The Green-Schroll set is the set of simple modules that **get rain!**

Characterization of quasi-hereditary algebras

Theroem 2

A Nakayama algebra A is quasi-hereditary if and only if $\mathcal{X}_A \neq \emptyset$.

Hereditary algebra	Quasi-hereditary algebra	Not quasi-hereditary
⇓	⇓	⇓
everyone gets rain	someone get rain	no one get rain

Examples

$$A = KA_n/I$$ ✓

$$A = K\tilde{A}_4/< \alpha_2\alpha_1, \alpha_3\alpha_2 >$$ ✓

$$A = K\tilde{A}_4/< \alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_4\alpha_3, \alpha_1\alpha_4 >$$ ✗

$$A = K\tilde{A}_4/< \alpha_1\alpha_4\alpha_3\alpha_2\alpha_1 >$$ ✗

Theorem

Let A be a Nakayama algebra. The following statements are equivalent:

(1) A is quasi-hereditary.

(2) There is a simple module of projective dimension 2[2].

(3) There is a simple ordering of simple modules $v_1, \ldots, v_n$ such that for each i, the simple module v_i is not properly internal to $T_{\widehat{v_1+\cdots+v_{i-1}}}$[3].

(4) A is S –connected[4].

(5) $\mathcal{X}_A \neq \emptyset$

The beautiful condition (2), can be viewed as the analogy of the well-known fact that algebras of global dimension 2 are quasi-hereditary. Condition (3), is originally proved to be true for all monomial algebras, so it is a little complicated. Condition (4), is surprisingly smart, comparing to condition (2). Condition (5),the theorem introduced before, is a criterion not involving any algebraic concepts so far.

Reference

[2] Morio Uematsu and Kunio Yamagata. On serial quasi-hereditary rings. *Hokkaido Math. J.*, 19(1):165–174, 1990.

[3] Edward L. Green and Sibylle Schroll. On quasi-hereditary algebras. *Bull. Sci. Math.*, 157:102797, 14, 2019.

[4] Ren´e Marczinzik and Emre Sen. A new characterization of quasi-hereditary Nakayama algebras and applications. *Comm. Algebra*, 50(10):4481–4493, 2022.

$q(KQ_n/<g>) =?$

Theorem[5]

Let A be a Nakayama algebra with one generator g and n simple modules. Then

$$q(A) = \frac{2}{l_g} n!$$

By Theorem 1, $q(A) = 2\binom{l_g}{1}\binom{n}{l_g}(n - l_g)! = \frac{2}{l_g} n!.$

[4] Yue Hui Zhang and Li Yu. Counting quasi-hereditary orderings of finite dimensional algebras. *J. MATH. TECH.*, 16(3):9–11, 2000.

The subset of $\mathcal{X}$

$\mathcal{X}^i$

Let $\mathcal{X}^0$, $\mathcal{X}^1$, $\mathcal{X}^2$, respectively, be the subset of $\mathcal{X}$ whose elements do not belong to any hood, are either hooks or denouements of some hoods but not both, and are both hooks and denouements of some hoods, respectively.

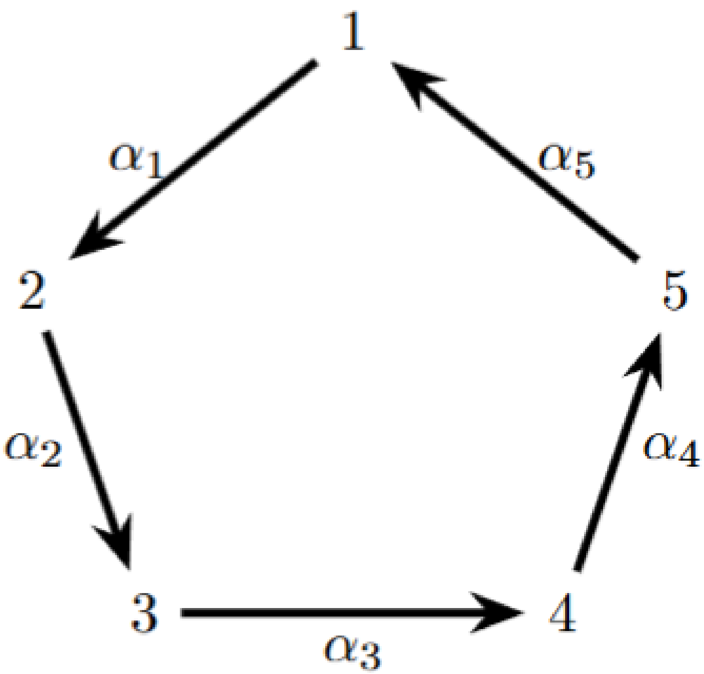
$q(KQ_n/<g_1, \dots, g_k>) =?$

Theroem 3

Let $A = KQ_n/I$ be a Nakayama algebra with Green-Schroll set $\mathcal{X}$. Then

$$q(A) = \frac{1}{n}[\sum_{x \in \mathcal{X}^0} q(A) + \sum_{x \in \mathcal{X}^1} q(KQ_n/I_{\hat{x}}) + \sum_{x \in \mathcal{X}^2} q(KQ_n/I_{\hat{x}})]$$

An example



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$1 \succcurlyeq 2 \succcurlyeq 3 \succcurlyeq 4 \succcurlyeq 5$ ✓

$5 \succcurlyeq 4 \succcurlyeq 3 \succcurlyeq 2 \succcurlyeq 1$ ✓

$4 \succcurlyeq 3 \succcurlyeq 2 \succcurlyeq 5 \succcurlyeq 1$ ✓

$$q(A) = \frac{1}{3} \times 5! = 40$$

Thank you for your attention!

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