Self-orthogonal τ-tilting modules and tilting modules

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AN OUTLINE

- Background.
- Preliminaries.
- Main results.
- Main references.

THE SETUP

- All algebras are finite-dimensional algebras over a field K, and all modules are finitely generated right modules.
- All tilting modules are classical tilting modules.
- $\mathbb{D}$ is the ordinary dual and $\tau = \mathbb{D}\mathrm{Tr}$.

Background

Tilting modules have been central in the representation theory of finite-dimensional algebras since 1970s. The recent definition of tilting modules is due to Happel and Ringel, see [HR] for details.

Definition 1.1 Let Λ be an algebra and $T \in \text{mod}\Lambda$. T is called a tilting module if the following are satisfied:

- $\text{Pd}\ T \leq 1$,
- $\text{Ext}^1_\Lambda(T, T) = 0$,
- $|T| = |\Lambda|$.

  The third item is equivalent to

- There is an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to 0$ with $T_i \in \text{add}T$.

Although tilting theorem and tilted algebras made the tilting theory fruitful, we want to emphasize the following two easy facts:

Facts 1.2 Let Λ be an algebra and $T \in \text{mod}\Lambda$ a tilting module.

- If $M \in \text{Fac}T = T^{\perp}$ and $\text{Hom}_{\Lambda}(T, M) = 0$, then $M = 0$.
- If $M \in \text{Fac}T = T^{\perp}$, then $\text{PdHom}_{\Lambda}(T, M) \leq \text{Pd}M$.

Background

Tilting modules are generalized to τ-tilting modules by Adachi, Iyama and Reiten in 2014 from the viewpoint of mutations. From then on, we get a method in computing tilting modules for algebras of finite representation type in theory. It is showed that τ-tilting theory is closely related to cluster-tilting theory and silting theory, see[AIR] for details.

Background

Definition 1.3 Let Λ be an algebra and $T \in \text{mod}\Lambda$.

- We call T a τ-rigid module if $\text{Hom}(T, \tau T)$=0,
- We call T a τ-tilting module if it is τ-rigid and $|T| = |\Lambda|$,
- We call T a support τ-tilting module if it is a τ-tilting module over $\Lambda/(e)$, where e is an idempotent.

Background

It is showed in [AIR] that tilting modules are faithful support τ-tilting modules and τ-tilting modules are sincere support τ-tilting modules.

Example 1.4 Let Λ be the finite dimensional algebra given by the quiver $Q : 1 \overset{a_1}{\to} 2 \overset{a_2}{\to} 3$ with the relation $a_1 a_2 = 0$. Then

- There are only two tilting modules $P(1) \oplus P(2) \oplus S(3)$, $P(1) \oplus P(2) \oplus S(2)$.
- There is a unique 2-tilting $P(1) \oplus P(2) \oplus S(1)$.
- There is a unique non-tilting τ-tilting module $P(1) \oplus S(1) \oplus S(3)$.

Background

- Classify τ-tilting modules over a given algebra, see [Mi] for preprojective algebras of Dynkin type, see [A] for Nakayama algebras, see [IZ] for the Auslander algebra of $k[x]/(x^n)$.
- Characterize algebras in terms of τ-tilting modules, see[DIJ].

In [XZZ], we showed that self-orthogonal τ-tilting modules admit very similar properties with tilting modules.

Proposition 1.6 Let A be an algebra and T a self-orthogonal τ-tilting module in mod$A$ with $B = \mathrm{End}_A T$. If $M \in \mathrm{Fac}T \cap T^{\perp}$, then $\mathrm{Pd}_B \mathrm{Hom}_A(T, M) \leq \mathrm{Pd}_A M$ holds.

Question 1.7 Is a self-orthogonal τ-tilting module a tilting module?

We should remark that the question is a special case of the Auslander-Reiten conjecture [AuR] as well as the Tachikawa's second conjecture [T] whenever the algebra is self-injective. Recently Wei also study this question in terms of Wakamatsu tilting modules.

Background

Auslander-Reiten Conjecture: Let Λ be an algebra and $M \in \text{mod}\Lambda$. If $\text{Ext}^i_\Lambda(M \oplus \Lambda, M \oplus \Lambda) = 0$ for $i \geq 1$, then M is projective.

Tachikawa's second Conjecture: Let Λ be a self-injective algebra and $M \in \text{mod}\Lambda$. If $\text{Ext}^i_\Lambda(M, M) = 0$ for $i \geq 1$, then M is projective.

Lemma 2.1 Let $0 \to Y \to T_0 \xrightarrow{f} X$ be an exact sequence in $\mathrm{mod}\Lambda$, where T is τ-rigid, and $f : T_0 \to X$ is a minimal right $\mathrm{add}T$-approximation of X. Then $\mathrm{Hom}_\Lambda(Y, \tau T) = 0$.

Lemma 2.2 Let $T \in \text{mod}\Lambda$ be a τ-rigid module. Then the following are equivalent.

- T is a τ-tilting module.
- $(\text{Fac}T, \text{Sub}\tau T)$ is a torsion pair.

Proposition 2.3 Let $T \in \mathrm{mod}\Lambda$ be a τ-tilting module. For any $M \in \mathrm{Fac}T$, there is an exact sequence $\cdots \to T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0$ with $T_i \in \mathrm{add}T$ and $\mathrm{Ker} f_i \in \mathrm{Fac}T$.

Preliminaries

Theorem 2.4 For any $M, N \in \mathrm{mod}\Lambda$, $\mathrm{Ext}^1_\Lambda(M, N) \simeq \mathbb{D}\overline{\mathrm{Hom}}_\Lambda(N, \tau M)$ holds.

Main results

Proposition 3.1 Let $T \in \text{mod}\Lambda$ be a τ-tilting module and let $M \in \text{mod}\Lambda$ satisfy $\text{Ext}^1_\Lambda(T, M) = 0$. Then $\text{Hom}_\Lambda(T, M) = 0$ holds if and only if $M = 0$.

Proof $\Rightarrow$ On the contrary, suppose that $M \neq 0$. Then the injective envelope $I^0(M)$ is not 0. Since T is a τ-tilting module, one gets a torsion pair $(\mathrm{Fac}T, \mathrm{Sub}\tau T)$ by Theorem 2.2. Then $\mathrm{Hom}_\Lambda(T, M) = 0$ implies that $M \in \mathrm{Sub}\tau T$. That is, $M \rightarrowtail (\tau T)^m$ for some integer $m \geq 1$. Since $\mathrm{Ext}^1_\Lambda(T, M) = 0$, by Theorem 2.4, one gets $\overline{\mathrm{Hom}}_\Lambda(M, \tau T) = 0$ which implies that the monomorphism $f : M \rightarrowtail (\tau T)^m$ factors through an injective module, and hence factors through the injective envelope $h : M \hookrightarrow I^0(M)$, that is, $f = gh$ for some $g : I^0(M) \to (\tau T)^m$. Since h is an essential monomorphism, one gets that g is a monomorphism, and hence $I^0(M)$ is a direct summand of τT. This is a contradiction.

Main results

Theorem 3.2 Let $T \in \mathrm{mod}\Lambda$ be a τ-tilting module. Then the following statements are equivalent.

- T is a tilting module.
- $\mathrm{Ext}^i_\Lambda(T, \mathrm{Fac}T) = 0$ for all $i \geq 1$
- $\mathrm{Ext}^i_\Lambda(T, \mathrm{Fac}T) = 0$ for $i = 2$.

Main results

Theorem 3.3 Let $T \in \mathrm{mod}\Lambda$. Then the following statements are equivalent.

- T is a tilting module.
- $\mathrm{Ext}^i_\Lambda(T, \mathrm{Fac}T) = 0$ for all $i \geq 1$ and $|T| = |\Lambda|$.
- $\mathrm{Ext}^i_\Lambda(T, \mathrm{Fac}T) = 0$ for $i = 1, 2$ and $|T| = |\Lambda|$.

Lemma 3.4 Let $T \in \mathrm{mod}\Lambda$ be a τ-tilting module with $\mathrm{Pd}_{\Lambda} T = d < \infty$. If $\mathrm{Ext}^i_{\Lambda}(T, T) = 0$ holds for all $i \geq 1$, then $\mathrm{Ext}^i_{\Lambda}(T, \mathrm{Fac}T) = 0$ for all $i \geq 1$.

Main results

Theorem 3.5 Let Λ be an algebra and $T \in \mathrm{mod}\Lambda$ a τ-tilting module of finite projective dimension. Then T is a tilting module if and only if $\mathrm{Ext}^i_\Lambda(T,T) = 0$ for all $i \geq 1$.

We should remark that a self-orthogonal support τ-tilting modules need not be partial tilting modules. We give an example to illustrate it.

Example 3.6 Let Λ be the finite dimensional algebra given by the quiver $Q : 1 \overset{a_1}{\to} 2 \overset{a_2}{\to} 3$ with the relation $a_1 a_2 = 0$. Then $S(1) = 1$ is an injective support τ-tilting module with projective dimension 2 and hence not a partial tilting module.

Theorem 3.7[CLZZ] Let Λ be an algebra of finite representation type and T a τ-tilting module in modΛ. Then T is a tilting module if and only if $\mathrm{Ext}^i_\Lambda(T, T) = 0$ for all $i \geq 1$.

We remark that Prof. Jiaqun Wei also gets Theorem 3.8 independently by using Wakamatsu tilting modules. We also remark that Prof. Wen Chang give a positive answer to the question over gentle algebras by using the geometric model.

Main references

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Thanks

Thanks for your attention!!!