An Introduction to Module Factorizations

Xiao-Wu Chen (陈小伍)
University of Science and Technology of China, Hefei, PR China

ICRA 21, organized by
Shanghai Jiao Tong University (SJTU), Shanghai, PR China
July 31–August 9, 2024

The highlight

Matrix factorizations due to David Eisenbud are very classical in singularity theory [Eisenbud 1980]. *Module factorizations* are their natural extensions, and might be of general interest.

The plan

- Matrix factorizations
- Module factorizations
- The main results

Section I

- Matrix factorizations
- Module factorizations
- The main results

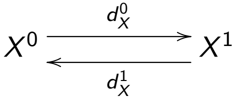
Let S be a commutative ring, and $\omega \in S$.

Definition (Eisenbud)

A *matrix factorization* of ω over S is a quadruple $X = (X^0, X^1; d_X^0, d_X^1)$, where X^i free S-modules of finite rank (or, finitely generated projective S-modules), and $d_X^0 \colon X^0 \to X^1$, $d_X^1 \colon X^1 \to X^0$ are homomorphisms satisfying

$$d_X^1 \circ d_X^0 = \omega \mathrm{Id}_{X^0} \text{ and } d_X^0 \circ d_X^1 = \omega \mathrm{Id}_{X^1}.$$

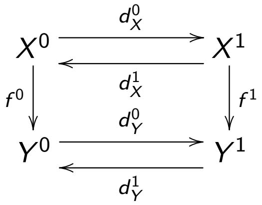
It is visualized as the following diagram.



1. In nice situation, $\mathrm{rank}(X^0) = \mathrm{rank}(X^1)$. By choosing bases for X^i, we obtain a pair of square matrices D^0 and D^1, such that
$$D^1 D^0 = \omega I_n = D^0 D^1.$$
This justifies the terminology.
2. When $n = 1$, a genuine factorization $\omega = D^1 D^0$ in S. Therefore, matrix factorizations generalize element factorizations.
3. Assume that $\omega = \det(A)$ for some matrix $A \in M_n(S)$. Then we have a canonical matrix factorization
$$AA^{\mathrm{ad}} = \omega I_n = A^{\mathrm{ad}} A.$$

Morphisms

A morphism $(f^0, f^1)\colon X \to Y$ between matrix factorizations is given by the following commutative diagram.

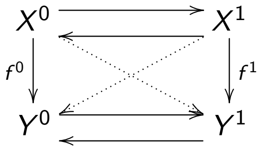


We form the category $\mathrm{MF}(S;\omega)$; naturally an exact category in the sense of Quillen.

Null-homotopic morphisms

A morphism $(f^0, f^1)\colon X \to Y$ between matrix factorizations is null-homotopic, if there are $h^0\colon X^0 \to Y^1$ and $h^1\colon X^1 \to Y^0$ such that

$$f^0 = d_Y^1 \circ h^0 + h^1 \circ d_X^0 \text{ and } f^1 = d_Y^0 \circ h^1 + h^0 \circ d_X^1.$$



Then we have the stable category $\underline{\mathrm{MF}}(S;\omega)$; it is naturally triangulated.

Modulo ω

Set $\overline{S} = S/(\omega)$. For each S-module M, set $\overline{M} = M/\omega M$.

Lemma (Eisenbud)

For a matrix factorization $X = (X^0, X^1; d_X^0, d_X^1)$, we have a (periodic) complex of free $\overline{S}$-modules:

$$\cdots \longrightarrow \overline{X^0} \longrightarrow \overline{X^1} \longrightarrow \overline{X^0} \longrightarrow$$

When ω is regular, the complex $\overline{X}^*$ is acyclic; moreover, its dual $\mathrm{Hom}_{\overline{S}}(\overline{X}^*, \overline{S})$ is also acyclic.

Remark: Use the dual matrix factorization $\mathrm{Hom}_S(X, S)$.

Gorenstein projective modules

Let R be any two-sided noetherian ring.

Definition (Auslander-Bridger)

A unbounded complex P^* of finitely generated projective R-modules is *totally acyclic*, if both P^* and its dual $\mathrm{Hom}_R(P^*, R)$ are acyclic.

A finitely generated R-module M is *Gorenstein projective* if there exists a totally acyclic complex P^* with $Z^0(P^*) = M$.

Remark: (1) also called *modules of G-dimension zero* by Auslander-Bridger, *maximal Cohen-Macaulay modules*, *totally reflexive modules* ...

(2) P^* is called *complete resolution* of M; [Tate 1952]

Let R be a commutative noetherian local ring. By a *maximal Cohen-Macaulay module* M, we mean $\mathrm{depth}(M) = \dim(R)$.

Theorem (Auslander-Bridger)

Let R be a commutative local Gorenstein ring. Then we have $R\text{-Gproj} = \mathrm{MCM}(R)$.

The *singularity category* of a noetherian ring R is $\mathbf{D}_{\mathrm{sg}}(R) = \mathbf{D}^b(R\text{-mod})/\mathrm{per}(R)$ [Buchweitz 1986/Orlov 2004].

Theorem (Buchweitz)

The canonical triangle functor

$$R\text{-}\underline{\mathrm{Gproj}} \longrightarrow \mathbf{D}_{\mathrm{sg}}(R)$$

is fully faithful; if R is Gorenstein, it is an equivalence.

The cokernel functor

Assume that ω is regular. Then we have the zeroth cokernel functor

$$\mathrm{Cok}^0\colon \mathrm{MF}(S;\omega) \longrightarrow \overline{S}\text{-Gproj},\ X \mapsto \mathrm{Cok}(d_X^0\colon X^0 \to X^1).$$

Theorem (Eisenbud)

The zeroth cokernel functor induces a fully faithful triangle functor

$$\underline{\mathrm{MF}}(S;\omega) \stackrel{\sim}{\longrightarrow} \overline{S}\text{-}\underline{\mathrm{Gproj}}.$$

Moreover, if $\mathrm{gl.dim}(S) < \infty$*, it is an equivalence.*

This global version is due to Orlov.

Noncommutative version is due to Cassidy-Conner-Kirkman-Moore and Mori-Ueyama.

The main question

Question

What about the case $\mathrm{gl.dim}(S) = \infty$, *with* $\overline{S}$ *(noncommutative) complete intersections in mind? What about arbitrary Gorenstein projective modules in the sense of Enochs-Jenda?*

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Eisenbud-Peeva's work indicates that we should consider module factorizations

$$X^0 \underset{d_X^1}{\overset{d_X^0}{\rightleftarrows}} X^1$$

with X^1 projective and $X^0 \in \mathrm{MCM}(S)$; implicitly called *generalized matrix factorizations*.


From Irena Peeva's homepage:

Eisenbud and I have wondered, for many years, how to describe the eventual patterns in the minimal resolutions of modules over complete intersections of higher codimension. With the theory developed in

the research monograph **Minimal Free Resolutions over Complete Intersections**

and the paper **Layered resolutions of Cohen-Macaulay modules**

we believe we have found an answer.

Section II

- Matrix factorizations
- Module factorizations
- The main results

Let S be any commutative ring, $\omega \in S$ regular.

Definition

A *module factorization* of ω over S is a diagram

$$X^0 \underset{d_X^1}{\overset{d_X^0}{\rightleftarrows}} X^1$$

with X^i arbitrary S-modules, satisfying $d_X^1 \circ d_X^0 = \omega \mathrm{Id}_{X^0}$ and $d_X^0 \circ d_X^1 = \omega \mathrm{Id}_{X^1}$. We form the abelian category $\mathrm{F}(S;\omega)$. Its full subcategories

$$\mathrm{MF}(S;\omega) \subseteq \mathrm{PF}(S;\omega) \subseteq \mathrm{G}^0\mathrm{F}(S;\omega) \subseteq \mathrm{GF}(S;\omega)$$

Linear factorizations, Dyckerhoff-Murfet, also Bergh-Thompson.

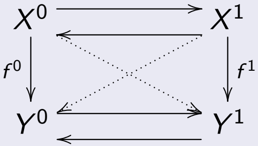
Categorically, Ballard-Deliu-Favero-Isik-Katzarkov and Bergh-Jorgensen.

A new homotopy relation

Definition

A morphism $(f^0, f^1)\colon X \to Y$ between module factorizations is *p-null-homotopic*, if there are $h^0\colon X^0 \to Y^1$ and $h^1\colon X^1 \to Y^0$ such that both factor through projective S-modules, and

$$f^0 = d_Y^1 \circ h^0 + h^1 \circ d_X^0 \text{ and } f^1 = d_Y^0 \circ h^1 + h^0 \circ d_X^1.$$



We form the stable category $\underline{\mathrm{F}}(S;\omega)$. It full subcategories

$$\underline{\mathrm{MF}}(S;\omega) \subseteq \underline{\mathrm{PF}}(S;\omega) \subseteq \underline{\mathrm{G^0F}}(S;\omega) \subseteq \underline{\mathrm{GF}}(S;\omega)$$

The stable category

Proposition

The category $\mathrm{GF}(S;\omega)$ *is Frobenius, and thus* $\underline{\mathrm{GF}}(S;\omega)$ *is triangulated.*

The proof: $\mathrm{F}(S;\omega) \simeq \Gamma$-Mod for some matrix ring Γ, and $\mathrm{GF}(S;\omega) \simeq \Gamma$-GProj by C.-Ren on *Frobenius functors*.

Section III

- Matrix factorizations
- Module factorizations
- The main results

Let S be any (possibly noncommutative) ring, and $\omega \in S$ regular (and normal).

Theorem

The zeroth cokernal functor induces a triangle equivalence

$$\underline{\mathrm{G}^0\mathrm{F}}(S;\omega) \xrightarrow{\sim} \overline{S}\text{-}\underline{\mathrm{GProj}},$$

which restricts to

$$\underline{\mathrm{PF}}(S;\omega) \xrightarrow{\sim} \overline{S}\text{-}\underline{\mathrm{GProj}}^{<+\infty},$$

RHS=the underlying S-module has finite projective dimension.

The proof: the hard part $\mathrm{Cok}^0(X)$ is Gorenstein projective over $\overline{S}$.

Related: Bahlekeh-Fotouhi-Nateghi-Salarian a finite version independently; Sun-Zhang on N-fold module factorizations.

A recollement

Theorem

There is an explicit recollement

$$\xymatrix{\underline{\mathrm{G^0F}}(S;\omega) \ar[rr]|{\mathrm{inc}} && \underline{\mathrm{GF}}(S;\omega) \ar@/_2pc/[ll]_{\mathrm{inc}_\lambda} \ar@/^2pc/[ll]^{\mathrm{inc}_\rho} \ar[rr]|{\mathrm{pr}^1} && \overline{S}\text{-}\underline{\mathrm{GProj}} \ar@/_2pc/[ll]_{\theta^1} \ar@/^2pc/[ll]^{\theta^0}}$$

Consequently, we have

$$\underline{\mathrm{GF}}(S;\omega)/\{\textit{trivial module factorizations}\} \simeq \overline{S}\text{-}\underline{\mathrm{GProj}}.$$

An example

Let G be a finite group, p a prime number.

$$\mathbb{F}_pG\text{-}\underline{\mathrm{Mod}} \longrightarrow \underline{\mathrm{GF}}(\mathbb{Z}G;p) \xrightarrow{\mathrm{pr}^1} \mathbb{Z}G\text{-}\underline{\mathrm{GProj}}$$

$$\mathbb{F}_pG\text{-}\underline{\mathrm{Mod}} \longleftarrow \underline{\mathrm{GF}}(\mathbb{Z}G;p), \quad \mathbb{F}_pG\text{-}\underline{\mathrm{Mod}} \xleftarrow{\widetilde{\mathrm{Cok}}^0} \underline{\mathrm{GF}}(\mathbb{Z}G;p), \quad \underline{\mathrm{GF}}(\mathbb{Z}G;p) \xleftarrow{\theta^1} \mathbb{Z}G\text{-}\underline{\mathrm{GProj}}, \quad \underline{\mathrm{GF}}(\mathbb{Z}G;p) \xleftarrow{\theta^0} \mathbb{Z}G\text{-}\underline{\mathrm{GProj}}$$

Remark: $\mathbb{Z}G$-$\underline{\mathrm{GProj}}$=$G$-lattices.

Thank you for your attention!

谢谢大家!

http://home.ustc.edu.cn/~xwchen