# The $\lambda$-pure singularity categories on a Grothendieck category

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# Outline

# Introduction

- The derived category is the smallest additive category that quasi-isomorphisms become isomorphisms, it is an important and complete set of theories.
- Neeman[5] considered the derived category of exact categories: Let $(\mathcal{A}, \mathcal{E})$ be an exact category, the derived category of $(\mathcal{A}, \mathcal{E})$, denoted by $\mathbf{D}(\mathcal{A})$, is the quotient of homotopy category $\mathbf{K}(\mathcal{A})$ by a thick subcategory (which is a full subcategory of $\mathbf{K}(\mathcal{A})$ consisting of the acyclic complexes).

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- Such as: Gao-Zhang introduced Gorenstein derived category; Asadollahi *et.al.* explored the relative derived category with respect to a contravariantly finite subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$; Zheng-Huang investigated the pure one in the module category.
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**Singularity Category**

Assume that $\mathcal{A}$ is an abelian category with enough projective objects. Denote by $\mathbf{K}^b(\mathcal{P})$ the subcategory of $\mathbf{K}^b(\mathcal{A})$ consisting of all complexes of projective objects. Then by [7, Lemma 5.1.10], it is a triangulated subcategory of the bounded derived category $\mathbf{D}^b(\mathcal{A})$. And it is also a thick one by Buchweitz. Thus, the Verdier quotient category $\mathbf{D}^b(\mathcal{A})/\mathbf{K}^b(\mathcal{P})$, in fact, is the usual singularity category of the abelian category $\mathcal{A}$, denoted by $\mathbf{D}^b_{sg}(\mathcal{A})$.

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- What we concerned is $\lambda$-pure exact structure, and we studied the $\lambda$-pure version derived category. As a continuation, we begin to explore the $\lambda$-pure singularity categories.

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# Preliminaries

We introduce some necessary notions and results which will be used repeatedly.

- Let $\lambda$ be an infinite regular cardinal. In what follows, $\mathcal{A}$ denotes a Grothendieck category and $\lambda$ is the least regular cardinal such that $\mathcal{A}$ is locally $\lambda$-presentable.
- Let $\mathbf{C}(\mathcal{A})$ and $\mathbf{K}(\mathcal{A})$ denote the category of complexes and homotopy category of $\mathcal{A}$, respectively.

For any $X \in \mathbf{C}(\mathcal{A})$, we write

$$X: \quad \cdots \to X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \to \cdots$$

A complex $X$ is acyclic if its cohomology

$$\mathrm{H}^i(X) := \ker d_X^i / \mathrm{Im} d_X^{i-1} = 0$$

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- Let $f : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism in $\mathbf{C}(\mathcal{A})$. Then $f$ is called a quasi-isomorphism, if it induces the isomorphism $\mathrm{H}^n(f) : \mathrm{H}^n(\boldsymbol{X}) \cong \mathrm{H}^n(\boldsymbol{Y})$ for each integer $n$.
- The mapping cone of a morphism $f : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{C}(\mathcal{A})$ is the complex $\mathrm{Cone}(f) := \boldsymbol{Z}$ with component $Z^n = X^{n+1} \bigoplus Y^n$ and differential $d_{\boldsymbol{Z}}^n = \begin{bmatrix} -d_{\boldsymbol{X}}^{n+1} & 0 \\ f^{n+1} & d_{\boldsymbol{Y}}^n \end{bmatrix}$. We know $f$ is a quasi-isomorphism if and only if $\mathrm{Cone}(f)$ is acyclic.

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Recall that an object $A \in \mathcal{A}$ is $\lambda$-presentable [1, Definition1.13(2)] if the functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ commutes with $\lambda$-directed colimits in $\mathcal{A}$. The following are some of our previous study on $\lambda$-pure derived category in [6].

**Definition 2.1**

A short exact sequence $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is $\lambda$-pure, if for any $\lambda$-presentable object $F$, $\mathrm{Hom}_{\mathcal{A}}(F, \xi)$ is an exact sequence. Naturally, in this case, we say that $f$ is a $\lambda$-pure monomorphism, and $g$ is a $\lambda$-pure epimorphism.

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**Definition 2.2**

An object $X$ in $\mathcal{A}$ is called $\lambda$-pure projective, if $X$ is projective with respect to $\lambda$-pure exact sequence. We denote by $\mathcal{PP}_\lambda$ the class of all $\lambda$-pure projective objects.

Note that the Grothendieck category $\mathcal{A}$ have enough $\lambda$-pure projective objects.

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**Definition 2.3**

A morphism $f : X \to Y$ in $\mathbf{C}(\mathcal{A})$ is called a $\lambda$-pure quasi-isomorphism if $\mathrm{Cone}(f)$ is $\lambda$-pure acyclic.

In fact, a morphism $f : X \to Y$ in $\mathbf{C}(\mathcal{A})$ is a $\lambda$-pure quasi-isomorphism if and only if $\mathrm{Hom}_{\mathcal{A}}(P, f)$ is a quasi-isomorphism for any $\lambda$-pure projective, or $\lambda$-presentable, object $P$.

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# Preliminaries

### $\lambda$-pure derived category

Let $\mathbf{K}_\lambda(\mathcal{A})$ be a subcategory of $\mathbf{K}(\mathcal{A})$ consisting of all $\lambda$-pure acyclic complexes, then it is thick. Naturally, the corresponding Verdier quotient is the $\lambda$-pure derived category of $\mathcal{A}$, that is, $\mathbf{D}_\lambda(\mathcal{A}) := \mathbf{K}(\mathcal{A})/\mathbf{K}_\lambda(\mathcal{A})$. Moreover, $\mathbf{D}^*_\lambda(\mathcal{A}) := \mathbf{K}^*(\mathcal{A})/\mathbf{K}^*_\lambda(\mathcal{A})$, where $* \in \{b, -, +\}$.

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**Theorem 2.4**

*There exists a triangle equivalence* $\mathbf{D}^{-}_{\lambda}(\mathcal{A}) \simeq \mathbf{K}^{-}(\mathcal{PP}_{\lambda})$. *In particular,* $\mathbf{D}^{b}_{\lambda}(\mathcal{A}) \simeq \mathbf{K}^{-,b}_{\lambda}(\mathcal{PP}_{\lambda})$.

Where $\mathbf{K}^{-,b}_{\lambda}(\mathcal{PP}_{\lambda})$ is the full subcategory of $\mathbf{K}^{-}(\mathcal{PP}_{\lambda})$ consisting of all complexes $X$ such that $\mathrm{Hom}_{\mathcal{A}}(F, X)$ is with bounded cohomology for any $\lambda$-presentable (more generally, $\lambda$-pure projective) object $F$. In fact, we say a complex $X \in \mathbf{K}^{-,b}_{\lambda}(\mathcal{PP}_{\lambda})$, if there exists an appropriate integer $n$, such that $X$ is $\lambda$-pure acyclic in degree $\leq n$.

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## Definition 2.5

The complex $\boldsymbol{X} \in \mathbf{D}_{\lambda}(\mathcal{A})$ is said to be of $\lambda$-pure projective dimension at most $n$, written by $\mathrm{p}_{\lambda}.\mathrm{pd}_{\mathcal{A}}\boldsymbol{X} \leq n$, if there exists a $\lambda$-pure quasi-isomorphism $\boldsymbol{P} \to \boldsymbol{X}$, where $\boldsymbol{P}$ is a complex of $\lambda$-pure projective objects, such that $P^i = 0$ for any $i < -n$.

## Definition 2.6

The $\lambda$-pure global dimension of $\mathcal{A}$, written by $\mathrm{P}_{\lambda}.\mathrm{gldim}\mathcal{A}$, is the supremum of the $\lambda$-pure projective dimensions of all objects in $\mathcal{A}$. It can be expressed as

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# Fundamental results

**Definition 3.1**

The $\lambda$-pure singularity category is defined to be the Verdier quotient

$$\mathbf{D}^b_{\lambda\text{-}sg}(\mathcal{A}) := \mathbf{D}^b_\lambda(\mathcal{A})/\mathbf{K}^b(\mathcal{PP}_\lambda) \cong \mathbf{K}^{-,b}_\lambda(\mathcal{PP}_\lambda)/\mathbf{K}^b(\mathcal{PP}_\lambda).$$

**Proposition 3.2**

*The $\lambda$-pure singularity category $\mathbf{D}^b_{\lambda\text{-}sg}(\mathcal{A}) = 0$ if and only if the $\lambda$-pure global dimension $\mathrm{P}_\lambda.\mathrm{gldim}\mathcal{A}$ is finite.*

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# Fundamental results

- We set $\boldsymbol{P}_\lambda$ a class consisting of all the projective resolution of $\lambda$-pure projective objects, and $\langle \boldsymbol{P}_\lambda \rangle$ the triangulated subcategory of $\mathbf{K}^{-,b}(\mathcal{P})$ generated by $\boldsymbol{P}_\lambda$, i.e., $\langle \boldsymbol{P}_\lambda \rangle$ is the smallest triangulated subcategory of $\mathbf{K}^{-,b}(\mathcal{P})$ that contains $\boldsymbol{P}_\lambda$ and is closed with respect to direct summands.

**Theorem 3.3**

*Let $\mathcal{A}$ be a Grothendieck category with enough projective objects. Then the triangulated functor*

$$G : (\mathbf{K}_\lambda^{-,b}/\mathbf{K}^b)(\mathcal{P}\mathcal{P}_\lambda) \to \mathbf{K}^{-,b}(\mathcal{P})/\langle \boldsymbol{P}_\lambda \rangle$$

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- We study the $\lambda$-pure derived equivalence as well as the $\lambda$-pure singular equivalence.

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*Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck categories, then the triangle equivalence*

$$F : \mathbf{K}^{-}(\mathcal{PP}_{\lambda}(\mathcal{A})) \to \mathbf{K}^{-}(\mathcal{PP}_{\lambda}(\mathcal{B}))$$

*induces a $\lambda$-pure derived equivalence $\mathbf{D}^{b}_{\lambda}(\mathcal{A}) \simeq \mathbf{D}^{b}_{\lambda}(\mathcal{B})$, and it also induces a $\lambda$-pure singular equivalence $\mathbf{D}^{b}_{\lambda\text{-}sg}(\mathcal{A}) \simeq \mathbf{D}^{b}_{\lambda\text{-}sg}(\mathcal{B})$.*

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# Fundamental results

- Let $\mathcal{A}$ be an abelian category with enough projective objects. The classical Buchweitz-Happel Theorem and its inverse show that the singularity category is triangle equivalent to a stable category.

**Theorem 3.5**

*[2, 4.4.1][4] Let $\mathcal{A}$ be an abelian category with enough projective objects. Then the canonical functor*

$$F : \underline{\mathcal{G}(\mathcal{A})} \to \mathbf{D}^b_{sg}(\mathcal{A})$$

*is a fully faithful triangle functor. Moreover, F is a triangulated equivalent if and only if the Gorenstein projective dimension of each object in $\mathcal{A}$ is finite, where the "only if" part holds by Zhu[8].*

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- Chen, and Li-Huang generalized the above Theorem, and they got the relative version of Buchweitz-Happel Theorem, respectively. Moreover, Gao-Zhang considered the Gorenstein version. And Cao *et.al.* explored the pure version.
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- The general construction of classic Buchweitz-Happel Theorem is not feasible for $\lambda$-pure one since the definition of the Gorenstein category is ordinary, i.e., replacing projective objects with $\lambda$-pure projective ones in the definition of Gorenstein projective, it is actually still $\lambda$-pure projective object.
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Thanks for your time and attention!