Rigidity dimensions and the Euclidean algorithm

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1. Rigidity dimensions

- Λ: finite dimensional algebra

$$0 \to {}_\Lambda\Lambda \to I^0 \to I^1 \to \cdots : \text{minimal injective resolution}$$

- The dominant dimension of Λ is

$$\mathrm{dom.dim}(\Lambda) := \inf\{d \in \mathbb{N} \mid I^i \text{ is NOT projective}\} \cup \{\infty\}$$

- The global dimension of Λ is

$$\mathrm{gl.dim}(\Lambda) := \sup\{\mathrm{proj.dim}\, X \mid X \in \Lambda\text{-mod}\}$$

The representation dimension of Λ [Auslander, 1971]

$$\mathrm{rep.dim}(\Lambda) = \inf\{\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) \mid M \text{ is a gen-cogen.}\}$$

The higher representation dimension [Iyama, 2005]

$$\mathrm{rep.dim}_n(\Lambda) := \inf\left\{\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) \,\middle|\, \begin{array}{l} M \text{ is a gen-cogen and} \\ \mathrm{dom.dim}\,\mathrm{End}_\Lambda(M) \geqslant n+1 \end{array}\right\}$$

$\mathrm{rep.dim}(\Lambda) = \mathrm{rep.dim}_1(\Lambda)$ is always finite [Iyama, 2003]

$\mathrm{rep.dim}_n(\Lambda)$ can be ∞ when $n > 1$.

The Rigidity dimension of Λ [CFKKY, 2021]

$$\mathrm{rig.dim}(\Lambda) = \sup \left\{ \mathrm{dom.dim}\, \mathrm{End}_\Lambda(M) \;\middle|\; \begin{array}{l} M: \text{gen-cogen and} \\ \mathrm{gl.dim}\, \mathrm{End}_\Lambda(M) < \infty \end{array} \right\}$$

Basic Facts:

(1) $\mathrm{rep.dim}_n(\Lambda) < \infty$ if and only if $\mathrm{rig.dim}(\Lambda) \geqslant n+1$.

(2) If Λ has a maximal n-orthogonal module, then

$$\mathrm{rig.dim}(\Lambda) \geqslant n+2$$

(3) Λ: non-selfinjective

$$\mathrm{rig.dim}(\Lambda) \leqslant \mathrm{inj.dim}\, {}_\Lambda\Lambda \leqslant \mathrm{gl.dim}(\Lambda) + 1$$

H. X. Chen, M. Fang, O. Kerner, S. Koenig and K. Yamagata, *Rigidity dimension of algebras.* Mathematical Proceedings of the Cambridge Philosophical Society **170** (2021), 417-443.

(4) H: non-semisimple hereditary algebra.

$$\mathrm{rig.\,dim}\, H = 2$$

Trivial extension of H: type A_n (CFKKY), type D_n (Chen-Xing, 2022), type E (R. Marczinzik)

(5) For a module X, its rigidity degree is

$$\mathrm{rd}(X) := \sup\{d \geqslant 0 \mid \mathrm{Ext}^i(M, M) = 0, \forall 0 < i \leqslant d\}.$$

(6) [Müller, 1968] M: gen-cogen.

$$\mathrm{dom.\,dim}\,\mathrm{End}(M) = \mathrm{rd}(M) + 2$$

(7) Λ: gendo-symmetric. Then

$$\mathrm{rig.\,dim}(\Lambda) \leqslant \mathrm{dom.\,dim}(\Lambda)$$

(8) $\bar{H}^* := HH^*(\Lambda)/\mathcal{N}$, $\mathcal{N}$: set of nilpotent elements.
If $\bar{H}^* \neq \bar{H}^0$ (not concerntrated in degree zero), define

$$\delta(\Lambda) := \min\{i > 0 \mid \bar{H}^i \neq 0\}.$$

Then $\mathrm{rig.\,dim}(\Lambda) \leqslant \delta(\Lambda) + 1$.

(9) Λ-$\underline{\text{mod}} \simeq \Gamma$-$\underline{\text{mod}}$ satisfying one of the following

- Λ, Γ have no nodes
- The stable equivalence is of Morita type
- The stable equivalence is a triangle equivalence.

Then $\operatorname{rig.dim}(\Lambda) = \operatorname{rig.dim}(\Gamma)$.

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- Find a gen-cogen M with $\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) < \infty$
- Calculate $\mathrm{dom.dim}\,\mathrm{End}_\Lambda(M) = 2 + \mathrm{rd}(M)$.
- Prove that $\mathrm{gl.dim}\,\mathrm{End}_\Lambda(N) = \infty$ for all gen-cogen N (if exists) with $\mathrm{rd}(N) > \mathrm{rd}(M)$.

2. Representation-finite selfinjective algebras

Question: Λ: fin dim rep-finite selfinjective $/K$, $\overline{K} = K$

$$\operatorname{rig.dim}(\Lambda) = ?$$

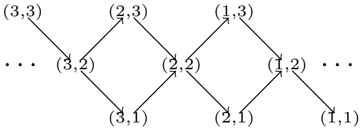
Λ: indec. non-simple rep-fin. selfinjective algebra.

The stable AR-quiver of Λ is

$$\Gamma_s(\Lambda) = \mathbb{Z}\Delta/G$$

Δ is a Dynkin diagram.
$G = \langle \tau^n \phi \rangle, n \geqslant 1$, $\phi \in \operatorname{Aut} \mathbb{Z}\Delta$ with a fixed vertex.

$\Delta = A_3$: $3 \to 2 \to 1$, $\mathbb{Z}\Delta$ is



$m_\Delta - 1 =$maximal length of nonzero paths in $k(\mathbb{Z}\Delta)$.

Δ	A_r	D_r	E_6	E_7	E_8
m_Δ	r	$2r-3$	11	17	29

s: the order of ϕ.

$(\Delta, n/m_\Delta, s)$ is called the type of Λ.

$\Lambda_{n,m}$: Nakayama algebra with n simples and radical length m.

The type of $\Lambda_{n,m}$ is $(A_{m-1}, \frac{n}{m-1}, 1)$.

$B(e, m)$: Brauer tree algebra with e simples and exceptional multiplicity m. Its type is $(A_{em}, 1/m, 1)$.

All possible types: (ADE)

- $(A_r, n/r, 1),\ r, n \in \mathbb{N};$
- $(A_{2p+1}, u, 2),\ p, s \in \mathbb{N};$
- $(D_r, u, 1),\ r, u \in \mathbb{N}, r \geqslant 4;$
- $(D_{3w}, u/3, 1),\ w, u \in \mathbb{N}, w \geqslant 2, 3 \nmid u;$
- $(D_r, u, 2),\ r, u \in \mathbb{N}, r \geqslant 4;$
- $(D_4, u, 3),\ u \in \mathbb{N};$
- $(E_r, u, 1),\ r = 6, 7, 8,\ u \in \mathbb{N};$
- $(E_6, u, 2),\ u \in \mathbb{N}.$

When Λ is of type (Δ_r, u, s), Λ has ru simple modules.

- Fact: For a module X over a selfinjective algebra, we have

$$\mathrm{rd}(X) = \mathrm{rd}(\tau X) = \mathrm{rd}(\Omega(X))$$

- Λ: indec. non-simple selfinjective algebra of finite rep. type
- Let X be an indecomposable module corresponding to the vertex (a, t) on its stable Auslander-Reiten quiver, then $\mathrm{rd}(X)$ depends only on t, we write $\mathrm{rd}(t)$ for $\mathrm{rd}(X)$.
- Question: $\mathrm{rd}(t) =?$

3. Weighted Fibonacci sequences

Given a sequence of integers $k_1, k_2, \cdots, k_r$, we define

$$\mathsf{F}_i := \begin{cases} 0, & i = -1 \\ 1, & i = 0 \\ k_i \mathsf{F}_{i-1} + \mathsf{F}_{i-2}, & i = 1, \cdots, r. \end{cases}$$

called the weighted Fibonacci sequence with weight sequence $k_1, \cdots, k_r$.

Given two positive integers m, n, the Euclidean algorithm gives

$$m = k_0 n + s_1, \quad n = k_1 s_1 + s_2, \cdots, s_d = k_{d+1} s_{d+1} + s_{d+2}, \quad s_{d+2} = 0$$

The weighted Fibonacci sequence determined the sequence $k_1, \cdots, k_{d+1}$ is denoted by $\mathsf{F}(m, n)$.

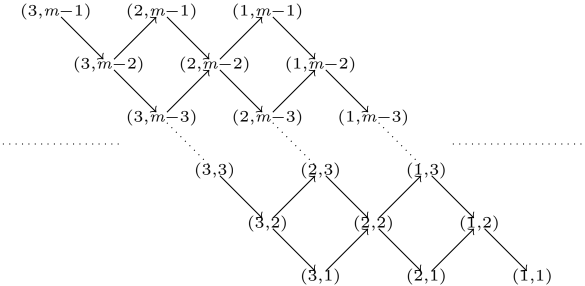
Example

$m = 32, n = 18$, set $s_0 = n$

i	0	1	2	3	4
k_i	1	1	3	2	
s_i	18	14	4	2	0
F_i	1	1	4	9	

4. Rigidity degrees of indecomposables

Type A: Suppose Λ is of type $(A_{m-1}, \frac{n}{m-1}, 1)$



$\omega(a,t) = (a+t, m-t)$, $\mathrm{rd}(t) = \mathrm{rd}(m-t)$.

Theorem (H-Yin 2024)

Suppose Λ *is of type* $(A_{m-1}, \frac{n}{m-1}, 1), m \geqslant 2$.

$\mathsf{F} = \mathsf{F}(m, n)$: *the weighted Fibonacci sequence determined by* m, n.

For $0 < t \leqslant m/2$, *we have*

$$\mathrm{rd}(t) = \begin{cases} 2\mathsf{F}_l - 1, & s_{l+1} < t < s_l, l < d+1 \text{ is even} \\ & \text{or } l = d+1 \\ 2\mathsf{F}_l, & s_{l+1} \leqslant t \leqslant s_l, l < d+1 \text{ is odd} \\ 2(\mathsf{F}_{d+1} - \mathsf{F}_d), & d+1 \text{ is odd}, t = s_{d+1} \leqslant m/2 \end{cases}$$

Wei Hu, Xiaojuan Yin *Rigidity degrees of indecomposable modules over representation-finite self-injective algebras.* J Pure Appl Algebra **228**(2024) 107498.

Example. $m = 32, n = 18,\ d = 2$

i	0	1	2	3	4
s_i	18	14	4	2	0
F_i	1	1	4	9	

range of t	(0,2)	2	(2,4)	[4,14]	(14,16]
$\mathrm{rd}(t)$	$2\mathsf{F}_3 - 1$	$2(\mathsf{F}_3 - \mathsf{F}_2)$	$2\mathsf{F}_2 - 1$	$2\mathsf{F}_1$	$2\mathsf{F}_0 - 1$
	17	10	7	2	1


31
1

Theorem (H-Yin 2024)

Suppose that Λ *is of type* (A_{m-1}, u, s). *Set*

$$M = m, \quad N = u(m-1), \text{ when } s = 1, \text{ and}$$

$$M = u(m-1) + m/2, \quad N = 2u(m-1), \text{ when } s = 2.$$

$\mathsf{F} = \mathsf{F}(M, N)$: *the weighted Fibonacci sequence determined by* M, N.
For $t \leqslant m/2$, *we have*

$\mathrm{rd}(t)$	*condition*
$\frac{2}{s}\mathsf{F}_l - 1$,	$s_{l+1} < t < s_l$, l *is even, or* $l = d+1$;
$\frac{2}{s}\mathsf{F}_l$,	$s_{l+1} \leqslant t \leqslant s_l$, $l < d+1$ *is odd;*
$\frac{2}{s}(\mathsf{F}_{d+1} - \mathsf{F}_d)$,	d *is even and* $t = s_{d+1} \leqslant m/2$.

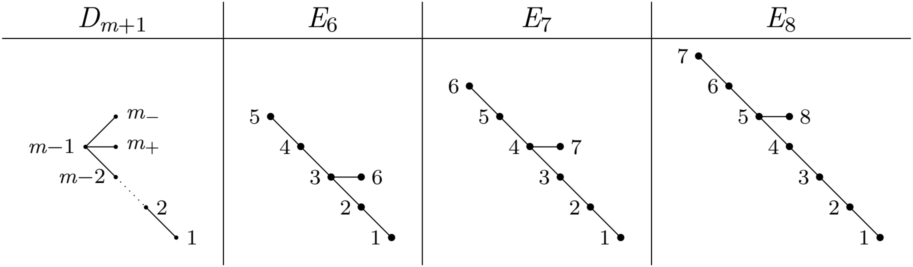
The Coxeter number of Δ:

$$h_\Delta = m_\Delta + 1$$

For $\Delta = A_{m-1}$, $m_\Delta = m - 1$ and $h_\Delta = m$.

Now for type (Δ, u, s), where Δ is of type D or E, we define

$$h^*_\Delta = h_\Delta/2, \quad n = um_\Delta, \quad \mathsf{F} = \mathsf{F}(h^*_\Delta, n)$$



Type (D_{m+1}, u, s) [H-Yin, 2024]

	vertex t	$\mathrm{rd}(t)$	condition
$s \neq 3$	$t < m$	$\mathsf{F}_l - 1$,	$s_{l+1} < t < s_l$, l is even, or $l = d+1$
		F_l,	$s_{l+1} \leqslant t \leqslant s_l$, $l < d+1$ is odd
		$\mathsf{F}_{d+1} - \mathsf{F}_d$,	$d+1$ is odd and $t = s_{d+1} < m$
$s \neq 3, m \geqslant n$	$t = m_\pm$	0	
$s \neq 3, m < n$	$t = m_\pm$	$\mathsf{F}_1 - 1$,	$m \mid n, \mathsf{F}_1 + n + s$ is odd
		$2\mathsf{F}_1 - 1$,	$m \mid n, \mathsf{F}_1 + n + s$ is even
		F_1,	$m \nmid n, \mathsf{F}_1 + n + s$ is even
		$\mathsf{F}_1 + \mathsf{F}_2$,	$m \nmid n, \mathsf{F}_1 + n + s$ is odd
$s = 3$	$t \neq 2$	$3\mathsf{F}_1 - 1$,	$u \equiv 0 \pmod 3$
		F_1,	$u \equiv 1 \pmod 3$
		$2\mathsf{F}_1$,	$u \equiv 2 \pmod 3$
	$t = 2$	$\mathsf{F}_1 - 1$,	$3 \mid u$
		F_1,	$3 \nmid u$

Type $(E_7, u, 1)$ [H-Yin, 2024]

$$\Delta = E_7, \quad h^*_\Delta = 9$$

$[u]_9$	0	1	2	3	4	5	6	7	8
$t=1$	F_1-1	F_1+3F_2	F_1+F_2	F_3-1	F_1	F_1+3F_2	F_2-1	F_1+F_2	F_1
$t=2$	F_1-1	F_1	F_1+F_2	F_1	F_1	F_1	$2F_1$	F_1	F_1
$t=3,4,5$	F_1-1	F_1	F_1	F_1	F_1	F_1	F_1	F_1	F_1
$t=6$	F_1-1	F_1+2F_2	F_1+3F_2	F_1	F_1+F_2	F_1	$2F_1$	$2F_1$	F_1
$t=7$	F_1-1	F_1+F_2	F_1	F_1	F_1	F_1	F_1	$2F_1$	F_1

Type $(E_8, u, 1)$ [H-Yin, 2024]

$$\Delta = E_8, \quad h^*_\Delta = 15$$

$[u]_{15}$	0	1	2	3	4	5	6	7
$t=1$	F_1-1	F_1+4F_2	F_3	F_1+2F_2	$F_1+F_2+F_3$	F_1	$2F_3$	F_1+F_2
$t=2$	F_1-1	F_1	F_1+F_2	F_1+F_2	F_1	F_1	F_1	F_1+F_2
$t=3,4,5,6$	F_1-1	F_1	F_1	F_1	F_1	F_1	F_1	F_1
$t=7$	F_1-1	F_1+2F_2	F_1+2F_2	F_1	F_1+F_2	F_1	F_1	F_1+F_2
$t=8$	F_1-1	F_1+F_2	F_1	F_1	F_1	F_1	F_1	F_1

$[u]_{15}$	8	9	10	11	12	13	14
$t=1$	F_1+F_2	F_1	$2F_1$	F_1+F_2	$2F_1$	$3F_1$	F_1
$t=2$	F_1	F_1	F_1	F_1+F_2	$2F_1$	F_1	F_1
$t=3,4,5,6$	F_1	F_1	F_1	F_1	F_1	F_1	F_1
$t=7$	F_1	F_1	$2F_1$	F_1	$2F_1$	$2F_1$	F_1
$t=8$	F_1	F_1	F_1	F_1	F_1	$2F_1$	F_1

Type (E_6, u, s), $h^*_\Delta = 6$ [H-Yin, 2024]

$(E_6, u, s), s = 1, 2$

$[u]_6$	0	1	2	3	4	5
$t = 3$	$\mathsf{F}_1 - 1$	F_1	F_1	F_1	F_1	F_1
$t = 6$	$\mathsf{F}_1 - 1$	$\mathsf{F}_1 + \mathsf{F}_2$	F_1	F_1	$2\mathsf{F}_1$	F_1

$(E_6, u, 1)$, $\lfloor u/6 \rfloor$ is even
$(E_6, u, 2)$, $\lfloor u/6 \rfloor$ is odd

$[u]_6$	0	1	2	3	4	5
$t = 1, 5$	$\mathsf{F}_1 - 1$	$\mathsf{F}_1 + 2\mathsf{F}_2$	$\mathsf{F}_1 + \mathsf{F}_3$	F_1	$2\mathsf{F}_1$	$3\mathsf{F}_1$
$t = 2, 4$	$\mathsf{F}_1 - 1$	F_1	F_1	F_1	F_1	$2\mathsf{F}_1$

Type (E_6, u, s), $h^*_\Delta = 6$ [H-Yin, 2024]

$(E_6, u, 1)$, $\lfloor u/6 \rfloor$ is odd
$(E_6, u, 2)$, $\lfloor u/6 \rfloor$ is even

$[u]_6$	0	1	2	3	4	5
$t = 1, 5$	$2\mathsf{F}_1 - 1$	$\mathsf{F}_1 + 4\mathsf{F}_2$	F_1	$\mathsf{F}_1 + \mathsf{F}_2$	$2\mathsf{F}_1$	F_1
$t = 2, 4$	$2\mathsf{F}_1 - 1$	F_1	F_1	F_1	F_1	F_1

5. Global dimensions of endomorphism algebras

Endomorphism algebras of finite global dimension

Question: How to find gen-cogen M with $\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) < \infty$?

Lemma

Λ: *fin. dim. alg.,* M: *gen-cogen. of* Λ. $\Gamma := \mathrm{End}_\Lambda(M)$.
X: Λ-*module. If* $\mathrm{proj.dim}_\Gamma \mathrm{Hom}_\Lambda(M, X) < \infty$, *then*

$$\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M \oplus X) < \infty \Rightarrow \mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) < \infty$$

Theorem

Λ: *fin. dim. alg.*

$0 \to X \longrightarrow Y \to Z \to 0$: *AR-seq. over* Λ.

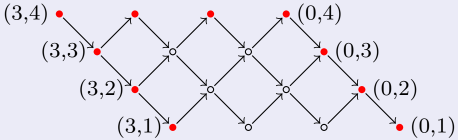
M: *gen-cogen. of* Λ *with* $Y \in \mathrm{add}(M)$.

If X *or* Z *belongs to* $\mathrm{add}(M)$, *then*

$$\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) < \infty \Leftrightarrow \mathrm{gl.dim}\,\mathrm{End}_\Lambda(M \oplus X \oplus Z) < \infty$$

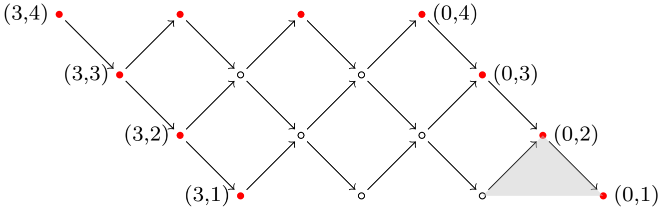
Example

Λ: selfinj. Nakayama alg. with 3 simples and Loewy length 4.
M: direct sum of modules corresp. to red vertices.

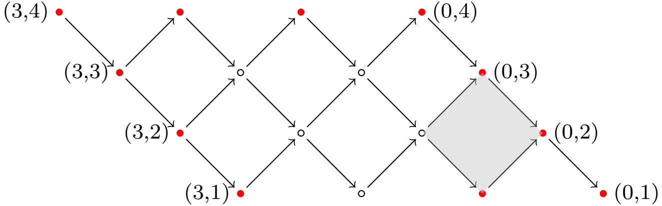


Then $\mathrm{gl.\,dim}\,\mathrm{End}_\Lambda(M) < \infty$.

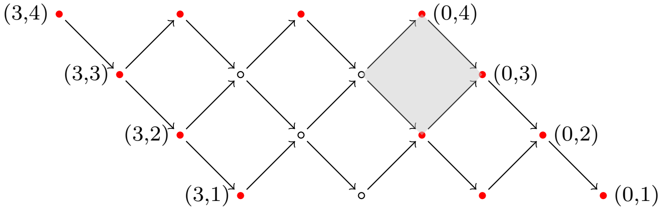
$M_0 := M$



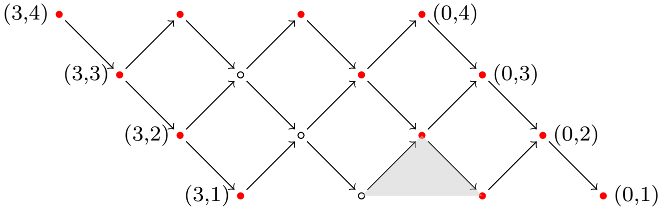
M_1



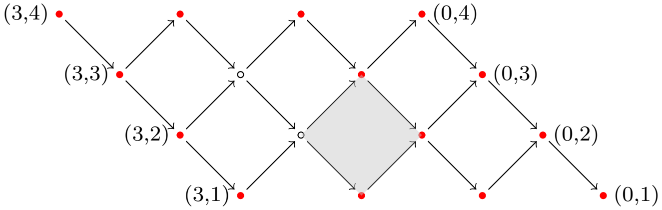
M_2



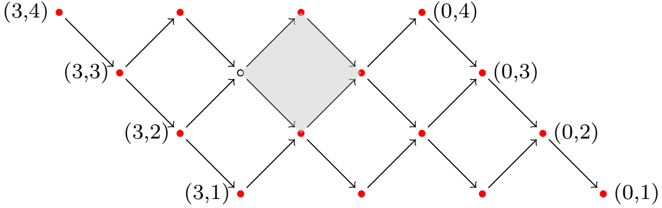
M_3



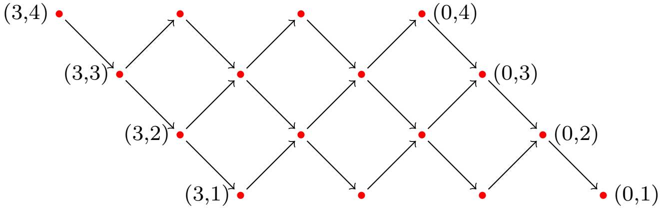
M_4



M_5



M_6, $\quad \mathrm{gl.dim}\,\mathrm{End}(M_6) = 2$. This implies $\mathrm{gl.dim}\,\mathrm{End}(M) < \infty$.



Proposition (Liu-Xi 2006)

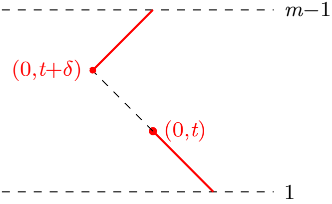
Λ: *selfinjective algebra*

X: Λ*-module*

Then

$$\begin{aligned} & \operatorname{gl.dim}\operatorname{End}_\Lambda(\Lambda \oplus X) \\ = & \operatorname{gl.dim}\operatorname{End}_\Lambda(\Lambda \oplus \Omega_\Lambda(X)) \\ = & \operatorname{gl.dim}\operatorname{End}_\Lambda(\Lambda \oplus \tau X) \end{aligned}$$

N_t^δ: gen-cogen over $\Lambda_{n,m}$, non-projective part:



Proposition

For $\Lambda_{n,m}$ with $m > n$. Suppose that $t + \delta \leqslant m$ and $\delta \leqslant n$. Then

$$\mathrm{gl.dim}\,\mathrm{End}(N_t^\delta) < \infty.$$

If $\delta = n$ and that $m - t - n \leqslant t \leqslant m/2$, then

$$\mathrm{rd}(N_t^n) = \mathrm{rd}(t).$$

6. Rigidity dimensions of selfinjective Nakayama algebras

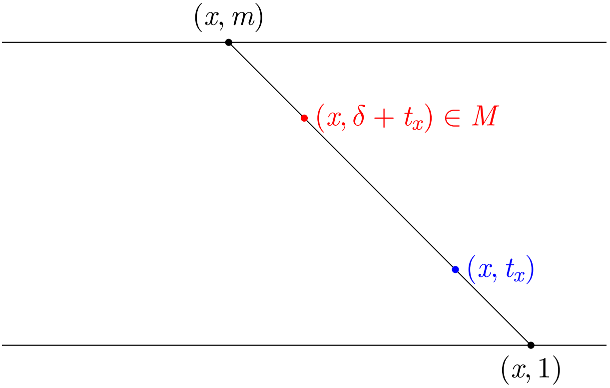
A criterion of infinite global dimension.

M: gen-cogen for $\Lambda := \Lambda_{n,m}$, $m > n$.

- Define $\delta = \max\{n, (k_0 - 1)n\}$
- For each $x \in \mathbb{Z}$, define

$$t_x = \min\{t \mid (x, t + \delta) \in M\}$$

- M is called δ-free if $(x, t_x) \notin M$ for all $x \in \mathbb{Z}$.



Proposition

$l < d+1$ *is odd,* $0 < s_{l+1} \leqslant m/2$

$M = N \oplus \Lambda$*: gen-cogen. with* $\mathrm{dom.dim}\,\mathrm{End}_\Lambda(M) \geqslant 2\mathsf{F}_l + 3$

If both M *and* $\Omega(N) \oplus \Lambda$ *are* δ*-free, then*

$$\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) = \infty.$$

Theorem (H-Yin, arXiv:2311.12697)

Suppose that Λ *is type* $(A_{m-1}, \frac{n}{m-1}, 1), m \geqslant n, m \geqslant 2$. *Then*

k_0	$\mathrm{rig.dim}(\Lambda)$	*condition*
1	3	$m = n \geqslant 2$
	$2\mathsf{F}_1 = 2n$	$m = n + 1$
	$2\mathsf{F}_1 + 1$	$s_1 \mid n, s_1 > 1$
	$2\mathsf{F}_1 + 3$	$k_2 = 1, s_2 - s_3 \geqslant 2$
	$2\mathsf{F}_1 + 2$	*else*
2	2	$s_1 = n - 1$
	3	$s_1 < n - 1$
$\geqslant 3$	2	

Theorem (H-Yin, arXiv:2311.12697)

Suppose that Λ *is type* $(A_{m-1}, \frac{n}{m-1}, 1), m \geqslant n, m \geqslant 2$. *Then*

k_0	$\mathrm{rig.dim}(\Lambda)$	*condition*
1	3	$m = n \geqslant 2$
	$2\mathsf{F}_1 = 2n$	$m = n + 1$
	$2\mathsf{F}_1 + 1$	$s_1 \mid n, s_1 > 1$
	$2\mathsf{F}_1 + 3$	$k_2 = 1, s_2 - s_3 \geqslant 2$
	$2\mathsf{F}_1 + 2$	*else*
2	2	$s_1 = n - 1$
	3	$s_1 < n - 1$
$\geqslant 3$	2	

Remark. For $m > n$, the above result fits the "mysterious" formula

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$$\mathrm{rig.dim}(\Lambda) = \left\lfloor \frac{2(m-1)}{m-n} \right\rfloor$$

Thanks!