Biset functors defined on categories

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The short story

- The biset category $\mathbb{B}$ has objects: finite categories, and morphisms: linear combinations of bisets. Bisets = profunctors = distributors. Disjoint unions of bisets are identified as their sum.
- Biset functors are linear functors $\mathbb{B} \to R$-mod.
- Historically the objects of $\mathbb{B}$ were just groups.
- Bisets encode restriction, induction, conjugation, inflation, deflation.
- Examples of biset functors: the Burnside ring, the representation ring, homology and cohomology (when representable bisets are used).
- Uses: refined information about representation rings; computation of cohomology; description of maps between classifying spaces; construction of transfer maps.

What is an action of a category on a set?

$\mathcal{C}$ is a finite category and Set denotes the category of finite sets.
A $\mathcal{C}$-set is a functor $\Omega : \mathcal{C} \to \mathrm{Set}$.

New definition! The Burnside ring of $\mathcal{C}$ is
$B(\mathcal{C})$ = the Grothendieck group finite $\mathcal{C}$-sets with relations
$\Theta = \Omega + \Psi$ if $\Theta \cong \Omega \sqcup \Psi$ as $\mathcal{C}$-sets.
The product of $\mathcal{C}$-sets is defined pointwise:
$(\Omega \cdot \Psi)(x) := \Omega(x) \times \Psi(x)$.

The poset $\mathcal{A}_2 = x < y$

Example

The $\sqcup$-indecomposable $\mathcal{A}_2$-sets have the form

$$\Omega_n := \{1, \ldots, n\} \to \{*\}, \quad n \geq 0.$$

The Burnside ring is

$$B(\mathcal{A}_2) = \mathbb{Z}\{\Omega_0, \Omega_1, \Omega_2, \ldots\} \cong \mathbb{Z}\mathbb{N}^{\times}_{\geq 0}$$

What are bisets for categories?

- If $\mathcal{C}$ and $\mathcal{D}$ are finite categories, a $(\mathcal{C}, \mathcal{D})$-biset is a set ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ with a left action of $\mathcal{C}$ and a right action of $\mathcal{D}$, so that these actions commute. This is the same as a set with an action of $\mathcal{C} \times \mathcal{D}^{\mathrm{op}}$.
- $A_R(\mathcal{C}, \mathcal{D})$ is the free R-module with the transitive finite $(\mathcal{C}, \mathcal{D})$-bisets as basis. We identify $[\Omega \sqcup \Psi] = [\Omega] + [\Psi]$
- Composition of bisets: ${}_{\mathcal{C}}\Omega_{\mathcal{D}} \circ {}_{\mathcal{D}}\Psi_{\mathcal{E}} = \Omega \times_{\mathcal{D}} \Psi$.
- The biset category $\mathbb{B}$ over a commutative ring R has objects finite groups, and homomorphisms $\mathrm{Hom}_{\mathbb{B}}(\mathcal{D}, \mathcal{C}) = A_R(\mathcal{C}, \mathcal{D})$.
- A biset functor is an R-linear functor $M : \mathbb{B} \to R$-mod.

Encoding homomorphisms and functors: special bisets

Recall: a biset functor M specifies: for each biset ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ an R-module homomorphism $M(\Omega) : M(\mathcal{D}) \to M(\mathcal{C})$.

For groups: restriction, corestriction are encoded by bisets ${}_H G_G$, ${}_G G_H$ where $H \leq G$. Inflation and deflation are encoded by bisets ${}_G Q_Q$, ${}_Q Q_G$ when Q is a quotient of G.

For categories: when $F : \mathcal{C} \to \mathcal{D}$ is a functor we get bisets ${}_{\mathcal{D}}\mathcal{D}_{F\mathcal{C}}$ and ${}_{\mathcal{C}F}\mathcal{D}_{\mathcal{D}}$

We have functors

$$\text{SimplicialComplexes} \to \text{SCat} \to \mathbb{B} \quad \text{or} \quad \mathbb{B}^{\text{op}}.$$

Examples of biset functors on categories

- The Burnside ring $B(\mathcal{C})$.

  In fact, $B(\mathcal{C})$ is the representable biset functor $\mathrm{Hom}_{\mathbb{B}}(1, -)$ evaluated at $\mathcal{C}$. This is because $\mathrm{Hom}_{\mathbb{B}}(1, \mathcal{C})$ is the Grothendieck group of $(\mathcal{C}, 1)$-bisets with respect to $\sqcup$, and these bisets are really the same as $\mathcal{C}$-sets.
  $B(-)$ is a projective biset functor.
- The Grothendieck group of finite dimensional $k\mathcal{C}$-modules with respect to direct sum decompositions $K_0(k\mathcal{C}, \oplus)$ is a biset functor.

Homology and cohomology of a category

The cohomology ring of $\mathcal{C}$ over R is $H^*(\mathcal{C}, R) := \mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, \underline{R})$.
The homology of $\mathcal{C}$ over R is $H_*(\mathcal{C}, R) := \mathrm{Tor}_*^{R\mathcal{C}}(\underline{R}, \underline{R})$.

These are isomorphic to the cohomology and homology of the nerve of $\mathcal{C}$.

A $\mathcal{C}$-set Ω is representable if $\Omega \cong \coprod \mathrm{Hom}(x_i, -)$, for some objects $x_i \in \mathcal{C}$.

Let $\mathbb{B}^{1,\mathsf{all}}$ be the subcategory of $\mathbb{B}$ obtained by using only bisets that are representable on the left, and $\mathbb{B}^{\mathsf{all},1}$ be the subcategory using bisets free on the right.

Cohomology of a category as a biset functor

Theorem (Bouc, Keller)

Hochschild homology $\mathcal{C} \mapsto HH_*(R\mathcal{C})$ *is a biset functor on* $\mathbb{B}^{1,all}$.

Theorem

Let R *be a field. Then* $\mathcal{C} \mapsto H^*(\mathcal{C}, R)$ *is a functor on* $\mathbb{B}^{all,1}$ *and* $\mathcal{C} \mapsto H_*(\mathcal{C}, R)$ *is a functor on* $\mathbb{B}^{1,all}$.

This provides a solution to the problem of finding a ‘corestriction’ map or ‘transfer’ in the cohomology of categories.

More theory

Theory of simple biset functors.

Internal tensor product: the direct product of categories $\mathcal{C} \times \mathcal{D}$
Unit: the category $\mathbf{1}$

Theorem

The biset category $\mathbb{B}$ is a rigid tensor category. The dual of a category $\mathcal{C}$ is $\mathcal{C}^{\mathrm{op}}$.

There is thus also an internal hom on $\mathbb{B}$.
A tensor product and internal hom can then be constructed on biset functors, giving rise to Green biset functors.

Structural properties of biset functors

Theorem (Coșkun, W., Snowden-W.)

Let R be a field of characteristic 0.

- *On finite groups, biset functors defined on $\mathbb{B}^{\mathbf{1},\mathbf{1}}$ are semisimple.*
- *On finite groups, biset functors defined on $\mathbb{B}$ are quasi-hereditary.*
- *Fix a prime p. Defined only on p-groups, finitely generated biset functors have finite length.*

Many questions

- What are the composition factors of the representation ring biset functor?
- What are the composition factors of the Burnside ring biset functor?
- Identify classes of categories so that, over a field of characteristic 0, the category of biset functors on $\mathbb{B}^{1,1}$ is semisimple.
- True or false?: If $\mathcal{C} \cong \mathcal{D}$ in $\mathbb{B}$ then the idempotent completions of $\mathcal{C}$ and $\mathcal{D}$ are equivalent.
- Are there simple biset functors S and finite categories $\mathcal{C}$ so that $\dim S(\mathcal{C})$ is infinite? Are there finite categories $\mathcal{C}$ and infinitely many simple biset functors S so that $S(\mathcal{C}) \neq 0$?
- Identify the Burnside rings of categories such as the orbit category of a finite group and relate it to the the Burnside ring of its fusion system.