NTNU | Institutt for matematiske fag

HIGHER τ-TILTING THEORY FOR NAKAYAMA ALGEBRAS

Work in progress with Endre S. Rundsveen

ICRA 21, Shanghai
August 2024

Aim

Aim

Investigate higher versions of τ-tilting theory inside a d-cluster tilting sub-category $\mathcal{C} \subseteq \operatorname{mod} \Lambda$.

Aim

Investigate higher versions of τ-tilting theory inside a d-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod} \Lambda$.

Specific example $\Lambda(n,l) = K\mathbb{A}_n/\langle$paths of length $l \geq 2\rangle$ where

$$\mathbb{A}_n : \qquad n \longrightarrow n-1 \longrightarrow \cdots \longrightarrow 3 \longrightarrow 2 \longrightarrow 1.$$

d-cluster tilting subcategories

Λ – a finite-dimensional algebra over a field $\mathbf{k}$

d-cluster tilting subcategories

Λ – a finite-dimensional algebra over a field $\mathbf{k}$

Definition (Iyama 2004)

Let $d \geq 1$ be an integer. A functorially finite subcategory $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ is a *d-cluster tilting subcategory* if

$$\begin{aligned} \mathcal{C} &= \{M \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i_\Lambda(M, \mathcal{C}) = 0 \text{ for all } 1 \leq i \leq d-1\} \\ &= \{M \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i_\Lambda(\mathcal{C}, M) = 0 \text{ for all } 1 \leq i \leq d-1\}. \end{aligned}$$

d-cluster tilting subcategories

Λ – a finite-dimensional algebra over a field $\mathbf{k}$

Definition (Iyama 2004)

Let $d \geq 1$ be an integer. A functorially finite subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is a *d-cluster tilting subcategory* if

$$\begin{aligned}\mathcal{C} &= \{M \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M,\mathcal{C}) = 0 \text{ for all } 1 \leq i \leq d-1\} \\ &= \{M \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(\mathcal{C},M) = 0 \text{ for all } 1 \leq i \leq d-1\}.\end{aligned}$$

d-Auslander–Reiten translations: $\tau_d = \tau\Omega^{d-1}$ and $\tau_d^- = \tau^-\Omega^{-(d-1)}$.

d-cluster tilting subcategories for $\Lambda(n, l)$

Theorem (V. 2018)

Let $\Lambda = \Lambda(n, l)$. There exists a d-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ if and only if there exists $p \geq 1$ such that

$$n = (p-1)\left(\frac{d-1}{2} l + 1\right) + \frac{l}{2}$$

and either

(i) $l = 2$, or

(ii) $l > 2$ and d and p are even.

τ-tilting theory

- Introduced by Adachi, Iyama and Reiten in 2013.

τ-tilting theory

- Introduced by Adachi, Iyama and Reiten in 2013.
- Generalise with respect to:

τ-tilting theory

- Introduced by Adachi, Iyama and Reiten in 2013.
- Generalise with respect to:

  1. Support τ-tilting pairs.

τ-tilting theory

- Introduced by Adachi, Iyama and Reiten in 2013.
- Generalise with respect to:

1. Support τ-tilting pairs.
2. Bijections with functorially finite torsion classes and 2-term silting complexes.

τ-tilting theory $\rightsquigarrow$ higher τ-tilting theory

τ-tilting theory $\rightsquigarrow$ higher τ-tilting theory

Definition (Adachi–Iyama–Reiten 2013)

A pair (M, P) with $M \in \operatorname{mod} \Lambda$ and $P \in \operatorname{proj} \Lambda$ is called *τ-rigid* if

$$\operatorname{Hom}_\Lambda(M, \tau(M)) = 0 \text{ and } \operatorname{Hom}_\Lambda(P, M) = 0.$$

τ-tilting theory $\rightsquigarrow$ higher τ-tilting theory

Definition (Adachi–Iyama–Reiten 2013)

A pair (M, P) with $M \in \operatorname{mod} \Lambda$ and $P \in \operatorname{proj} \Lambda$ is called *τ-rigid* if

$$\operatorname{Hom}_\Lambda(M, \tau(M)) = 0 \text{ and } \operatorname{Hom}_\Lambda(P, M) = 0.$$

Definition (Jacobsen–Jørgensen 2020)

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ be a d-cluster tilting subcategory. A pair (M, P) with $M \in \mathcal{C}$ and $P \in \operatorname{proj} \Lambda$ is called *τ_d-rigid* if

$$\operatorname{Hom}_\Lambda(M, \tau_d(M)) = 0 \text{ and } \operatorname{Hom}_\Lambda(P, M) = 0.$$

τ-tilting theory $\rightsquigarrow$ higher τ-tilting theory

τ-tilting theory $\rightsquigarrow$ higher τ-tilting theory

Definition-Proposition (Adachi–Iyama–Reiten 2013)

Let (M, P) be a τ-rigid pair over Λ. Then (M, P) is called a *support τ-tilting pair* if any of the following equivalent conditions holds.

τ-tilting theory $\rightsquigarrow$ higher τ-tilting theory

Definition-Proposition (Adachi–Iyama–Reiten 2013)

Let (M, P) be a τ-rigid pair over Λ. Then (M, P) is called a *support τ-tilting pair* if any of the following equivalent conditions holds.

(a) If $(M \oplus N, P)$ is a τ-rigid pair for some $N \in \operatorname{mod} \Lambda$, then $N \in \operatorname{add} M$.

Definition-Proposition (Adachi–Iyama–Reiten 2013)

Let (M, P) be a τ-rigid pair over Λ. Then (M, P) is called a *support τ-tilting pair* if any of the following equivalent conditions holds.

(a) If $(M \oplus N, P)$ is a τ-rigid pair for some $N \in \operatorname{mod} \Lambda$, then $N \in \operatorname{add} M$.

(b) If $\operatorname{Hom}_\Lambda(M, \tau(N)) = 0$, $\operatorname{Hom}_\Lambda(N, \tau(M)) = 0$ and $\operatorname{Hom}_\Lambda(P, N) = 0$ for some $N \in \operatorname{mod} \Lambda$, then $N \in \operatorname{add} M$.

τ-tilting theory $\rightsquigarrow$ higher τ-tilting theory

Definition-Proposition (Adachi–Iyama–Reiten 2013)

Let (M, P) be a τ-rigid pair over Λ. Then (M, P) is called a *support τ-tilting pair* if any of the following equivalent conditions holds.

(a) If $(M \oplus N, P)$ is a τ-rigid pair for some $N \in \operatorname{mod} \Lambda$, then $N \in \operatorname{add} M$.

(b) If $\operatorname{Hom}_\Lambda(M, \tau(N)) = 0$, $\operatorname{Hom}_\Lambda(N, \tau(M)) = 0$ and $\operatorname{Hom}_\Lambda(P, N) = 0$ for some $N \in \operatorname{mod} \Lambda$, then $N \in \operatorname{add} M$.

(c) If (M', P') is another τ-rigid pair, then $|M'| + |P'| \leq |M| + |P|$.

τ-tilting theory $\rightsquigarrow$ higher τ-tilting theory

Definition-Proposition (Adachi–Iyama–Reiten 2013)

Let (M, P) be a τ-rigid pair over Λ. Then (M, P) is called a *support τ-tilting pair* if any of the following equivalent conditions holds.

(a) If $(M \oplus N, P)$ is a τ-rigid pair for some $N \in \operatorname{mod} \Lambda$, then $N \in \operatorname{add} M$.

(b) If $\operatorname{Hom}_\Lambda(M, \tau(N)) = 0$, $\operatorname{Hom}_\Lambda(N, \tau(M)) = 0$ and $\operatorname{Hom}_\Lambda(P, N) = 0$ for some $N \in \operatorname{mod} \Lambda$, then $N \in \operatorname{add} M$.

(c) If (M', P') is another τ-rigid pair, then $|M'| + |P'| \leq |M| + |P|$.

(d) $|M| + |P| = |\Lambda|$.

Definition

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ be a d-cluster tilting subcategory and let (M, P) be a τ_d-rigid pair. Then (M, P) is called a

τ-tilting theory $\rightsquigarrow$ higher τ-tilting theory

Definition

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ be a d-cluster tilting subcategory and let (M, P) be a τ_d-rigid pair. Then (M, P) is called a

(a) *support τ_d-tilting pair* if whenever $(M \oplus N, P)$ is a τ_d-rigid pair for some $N \in \mathcal{C}$, then $N \in \operatorname{add} M$ and P is maximal (Zhou–Zhu 2023).

τ-tilting theory $\rightsquigarrow$ higher τ-tilting theory

Definition

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ be a d-cluster tilting subcategory and let (M, P) be a τ_d-rigid pair. Then (M, P) is called a

(a) *support τ_d-tilting pair* if whenever $(M \oplus N, P)$ is a τ_d-rigid pair for some $N \in \mathcal{C}$, then $N \in \operatorname{add} M$ and P is maximal (Zhou–Zhu 2023).

(b) *maximal τ_d-rigid pair* if whenever $\operatorname{Hom}_\Lambda(M, \tau_d(N)) = 0$, $\operatorname{Hom}_\Lambda(N, \tau_d(M)) = 0$ and $\operatorname{Hom}_\Lambda(P, N) = 0$ hold for some $N \in \mathcal{C}$, then $N \in \operatorname{add} M$ and P is maximal (Jacobsen–Jørgensen 2020).

τ-tilting theory $\rightsquigarrow$ higher τ-tilting theory

Definition

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ be a d-cluster tilting subcategory and let (M, P) be a τ_d-rigid pair. Then (M, P) is called a

(a) *support τ_d-tilting pair* if whenever $(M \oplus N, P)$ is a τ_d-rigid pair for some $N \in \mathcal{C}$, then $N \in \operatorname{add} M$ and P is maximal (Zhou–Zhu 2023).

(b) *maximal τ_d-rigid pair* if whenever $\operatorname{Hom}_\Lambda(M, \tau_d(N)) = 0$, $\operatorname{Hom}_\Lambda(N, \tau_d(M)) = 0$ and $\operatorname{Hom}_\Lambda(P, N) = 0$ hold for some $N \in \mathcal{C}$, then $N \in \operatorname{add} M$ and P is maximal (Jacobsen–Jørgensen 2020).

(c) *strongly maximal τ_d-rigid pair* if whenever (M', P') is another τ_d-rigid pair, we have $|M'| + |P'| \leq |M| + |P|$ (Rundsveen–V.).

Higher τ-tilting theory ($d > 1$)

Higher τ-tilting theory ($d > 1$)

(M, P) strongly maximal τ_d-rigid pair $\implies$ (M, P) support τ_d-tilting pair

Higher τ-tilting theory ($d > 1$)

(M, P) strongly maximal τ_d-rigid pair $\implies$ (M, P) support τ_d-tilting pair

No converse.

Higher τ-tilting theory ($d > 1$)

$$(M, P) \text{ strongly maximal } \tau_d\text{-rigid pair} \implies (M, P) \text{ support } \tau_d\text{-tilting pair}$$

No converse.

$$(M, P) \text{ maximal } \tau_d\text{-rigid pair} \implies (M, P) \text{ support } \tau_d\text{-tilting pair}$$

Higher τ-tilting theory ($d > 1$)

$$(M, P) \text{ strongly maximal } \tau_d\text{-rigid pair} \implies (M, P) \text{ support } \tau_d\text{-tilting pair}$$

No converse.

$$(M, P) \text{ maximal } \tau_d\text{-rigid pair} \implies (M, P) \text{ support } \tau_d\text{-tilting pair}$$

No converse in general (Zhou–Zhu 2023), converse holds if no module $N \in \mathcal{C}$ satisfies $\mathrm{Hom}_\Lambda(N, \tau_d(N)) \neq 0$.

Higher τ-tilting theory ($d > 1$)

$$(M, P) \text{ strongly maximal } \tau_d\text{-rigid pair} \implies (M, P) \text{ support } \tau_d\text{-tilting pair}$$

No converse.

$$(M, P) \text{ maximal } \tau_d\text{-rigid pair} \implies (M, P) \text{ support } \tau_d\text{-tilting pair}$$

No converse in general (Zhou–Zhu 2023), converse holds if no module $N \in \mathcal{C}$ satisfies $\mathrm{Hom}_\Lambda(N, \tau_d(N)) \neq 0$.

$$(M, P) \text{ strongly maximal } \tau_d\text{-rigid pair} \implies |M| + |P| \geq |\Lambda|$$

Higher τ-tilting theory ($d > 1$)

Theorem (Rundsveen–V.)

Let $d > 1$ and assume $\mathcal{C} \subseteq \operatorname{mod} \Lambda(n, l)$ is a d-cluster tilting subcategory. Let $M \in \mathcal{C}$ and $P \in \operatorname{proj} \Lambda$. Then

$$\begin{aligned}
(M, P) \text{ strongly maximal } \tau_d\text{-rigid} &\iff (M, P)\ \tau_d\text{-rigid and } |M| + |P| = |\Lambda| \\
&\implies (M, P) \text{ maximal } \tau_d\text{-rigid} \\
&\iff (M, P) \text{ support } \tau_d\text{-tilting}
\end{aligned}$$

Higher τ-tilting theory ($d > 1$)

Theorem (Rundsveen–V.)

Let $d > 1$ and assume $\mathcal{C} \subseteq \operatorname{mod} \Lambda(n,l)$ is a d-cluster tilting subcategory. Let $M \in \mathcal{C}$ and $P \in \operatorname{proj} \Lambda$. Then

$$\begin{aligned}(M,P) \text{ strongly maximal } \tau_d\text{-rigid} &\iff (M,P)\ \tau_d\text{-rigid and } |M|+|P|=|\Lambda| \\ &\implies (M,P) \text{ maximal } \tau_d\text{-rigid} \\ &\iff (M,P) \text{ support } \tau_d\text{-tilting}\end{aligned}$$

Proof via an explicit combinatorial classification of strongly maximal τ_d-rigid pairs.

Higher τ-tilting theory

Example

$n = 23, l = 4$:

4 3 2 1	5 4 3 2	6 5 4 3	7 6 5 4	8 7 6 5	9 8 7 6	10 9 8 7	11 10 9 8	12 11 10 9	13 12 11 10	14 13 12 11	15 14 13 12	16 15 14 13	17 16 15 14	18 17 16 15	19 18 17 16	20 19 18 17	21 20 19 18	22 21 20 19	23 22 21 20

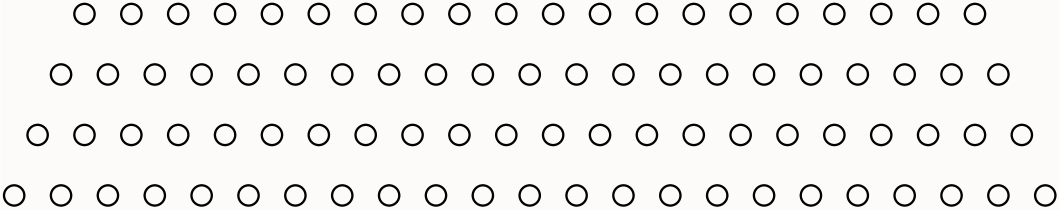
3 2 1	4 3 2	5 4 3	6 5 4	7 6 5	8 7 6	9 8 7	10 9 8	11 10 9	12 11 10	13 12 11	14 13 12	15 14 13	16 15 14	17 16 15	18 17 16	19 18 17	20 19 18	21 20 19	22 21 20	23 22 21

2 1	3 2	4 3	5 4	6 5	7 6	8 7	9 8	10 9	11 10	12 11	13 12	14 13	15 14	16 15	17 16	18 17	19 18	20 19	21 20	22 21	23 22

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23

Example

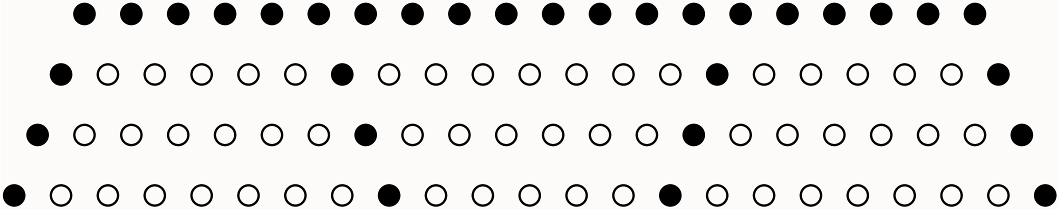
$n = 23$, $l = 4$:



Higher τ-tilting theory

Example

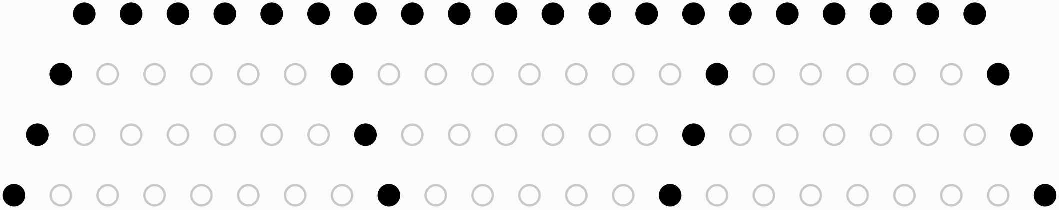
$n = 23$, $l = 4$, $d = 4$, $\mathcal{C}$:



Higher τ-tilting theory

Example

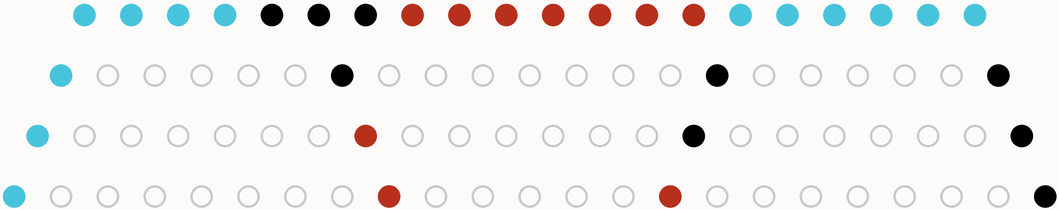
$n = 23$, $l = 4$, $d = 4$, $\mathcal{C}$:



Higher τ-tilting theory

Example

$n = 23$, $l = 4$, $d = 4$, $\mathcal{C}$, (M, P) strongly maximal τ_4-rigid pair:



d-torsion classes

$\mathcal{C} \subseteq \operatorname{mod} \Lambda$ – a d-cluster tilting subcategory

Definition (Jørgensen 2016)

We say that $\mathcal{U} \subseteq \mathcal{C}$ is a *d-torsion class* if for every $C \in \mathcal{C}$ there exists a d-extension

$$0 \longrightarrow U \xrightarrow{u} C \xrightarrow{c_0} C_1 \xrightarrow{c_1} \cdots \xrightarrow{c_{d-1}} C_d \longrightarrow 0$$

such that $U \in \mathcal{U}$ and, for every $U' \in \mathcal{U}$, the induced sequence

$$0 \longrightarrow \operatorname{Hom}_\Lambda(U', C_1) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_\Lambda(U', C_d) \longrightarrow 0$$

is exact.

d-torsion classes for $\Lambda(n, l)$

d-torsion classes for $\Lambda(n, l)$

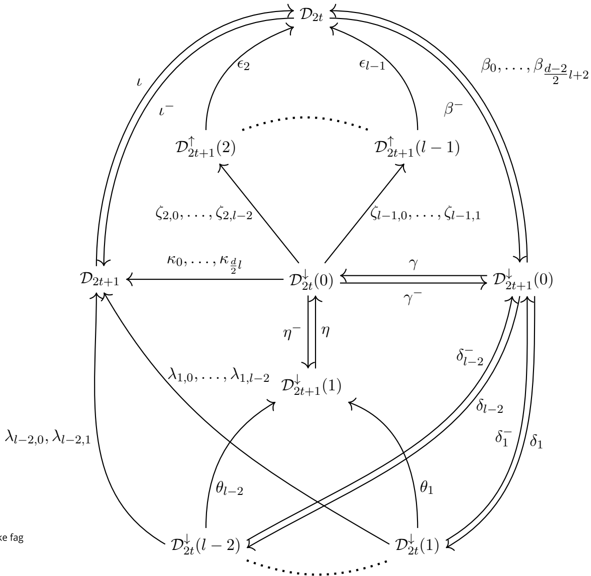
Theorem (Rundsveen-V.)

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda(n, l)$ be a d-cluster tilting subcategory.

There exists an explicit bijection between the set of directed paths in the directed multigraph G of length $p-1$ starting at an odd vertex and the set of d-torsion classes $\mathcal{U} \subseteq \mathcal{C}$.

d-torsion classes for $\Lambda(n, l)$

The graph G:

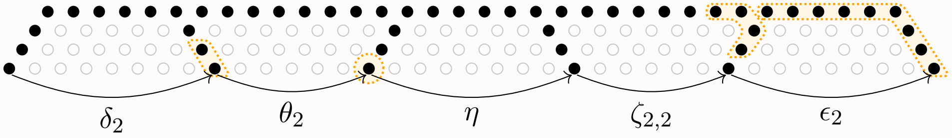


(+additional arrows for $d = 2$)

d-torsion classes for $\Lambda(n, l)$

Example

$n = 37$, $l = 4$, $d = 4$, $\mathcal{U}$ 4-torsion class:



d-torsion classes, strongly maximal τ_d-rigid pairs and $(d+1)$-silting complexes

Theorem (August–Haugland–Jacobsen–Kvamme–Palu–Treffinger)

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ be a d-cluster tilting subcategory. There exist maps

$$\left\{ \begin{array}{c} \text{functorially finite} \\ d\text{-torsion classes in } \mathcal{C} \end{array} \right\} \xrightarrow{\phi_d} \left\{ \begin{array}{c} \text{basic maximal } \tau_d\text{-rigid} \\ \text{pairs } (M,P) \text{ in } \mathcal{C} \text{ with} \\ |M| + |P| = |\Lambda| \end{array} \right\} \xrightarrow{\psi_d} \left\{ \begin{array}{c} \text{basic } (d+1)\text{-silting} \\ \text{complexes in } \mathrm{K}^b(\operatorname{proj} \Lambda) \end{array} \right\}.$$

where

- ϕ_d is injective but not surjective when $d > 1$.
- ψ_d is defined on $\operatorname{im}(\phi_d)$.

d-torsion classes, strongly maximal τ_d-rigid pairs and $(d+1)$-silting complexes

Theorem (August–Haugland–Jacobsen–Kvamme–Palu–Treffinger)

Let $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ be a d-cluster tilting subcategory. There exist maps

$$\left\{ \begin{array}{c} \text{functorially finite} \\ d\text{-torsion classes in } \mathcal{C} \end{array} \right\} \xrightarrow{\phi_d} \left\{ \begin{array}{c} \text{basic maximal } \tau_d\text{-rigid} \\ \text{pairs } (M,P) \text{ in } \mathcal{C} \text{ with} \\ |M| + |P| = |\Lambda| \end{array} \right\} \xrightarrow{\psi_d} \left\{ \begin{array}{c} \text{basic } (d+1)\text{-silting} \\ \text{complexes in } \mathrm{K}^b(\operatorname{proj}\Lambda) \end{array} \right\}.$$

where

- ϕ_d is injective but not surjective when $d > 1$.
- ψ_d is defined on $\operatorname{im}(\phi_d)$.

If $\Lambda = \Lambda(n,l)$, then we extend ψ_d to the set of all τ_d-rigid pairs with $|M| + |P| = |\Lambda|$.

Mutation graph

Mutation graph

We define two basic strongly maximal τ_d-rigid pairs to be *τ_d-mutations of each other* if they differ by exactly one indecomposable summand.

Mutation graph

We define two basic strongly maximal τ_d-rigid pairs to be *τ_d-mutations of each other* if they differ by exactly one indecomposable summand.

We also define the *mutation graph of strongly maximal τ_d-rigid pairs* of Λ as the graph having the strongly maximal τ_d-rigid pairs as nodes, and edges between two nodes if the pairs are τ_d-mutations of each other.

Mutation graph

Proposition (Rundsveen-V.)

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda(n,l)$ be a d-cluster tilting subcategory. Then the mutation graph of strongly maximal τ_d-rigid pairs of $\Lambda(n,l)$ is connected.

Mutation graph

Proposition (Rundsveen-V.)

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda(n,l)$ be a d-cluster tilting subcategory. Then the mutation graph of strongly maximal τ_d-rigid pairs of $\Lambda(n,l)$ is connected.

Contrary to the classical case, if $d > 1$, then the mutation graph of strongly maximal τ_d-rigid pairs is not regular.

Mutation graph

Proposition (Rundsveen-V.)

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda(n,l)$ be a d-cluster tilting subcategory. Then the mutation graph of strongly maximal τ_d-rigid pairs of $\Lambda(n,l)$ is connected.

Contrary to the classical case, if $d > 1$, then the mutation graph of strongly maximal τ_d-rigid pairs is not regular.

With one exception...

Mutation graph

Proposition (Rundsveen–V.)

Let $\mathcal{C} \subseteq \operatorname{mod}\Lambda(n,l)$ be a d-cluster tilting subcategory. Assume that the global dimension of $\Lambda(n,l)$ is also d.

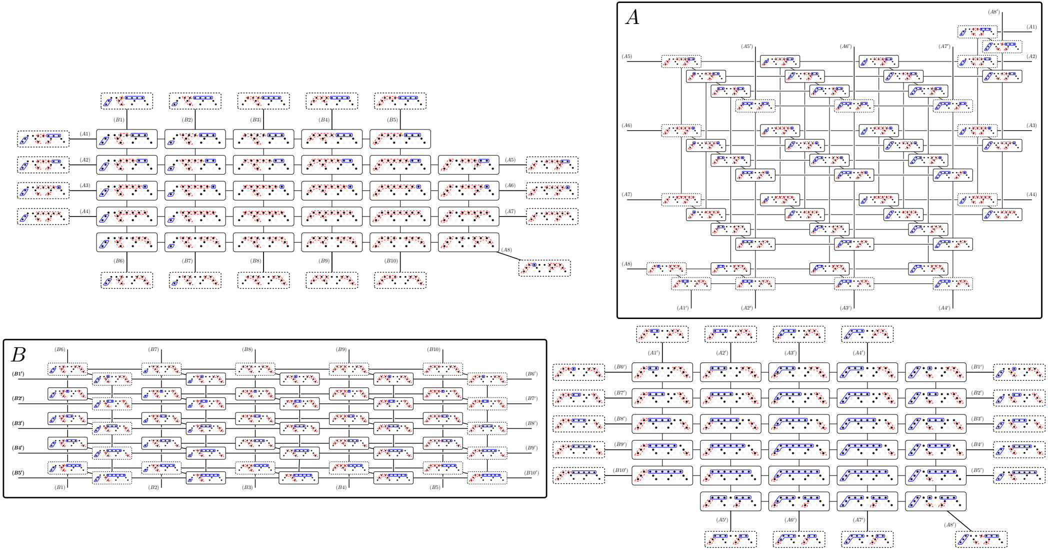
Mutation graph

Proposition (Rundsveen–V.)

Let $\mathcal{C} \subseteq \operatorname{mod}\Lambda(n,l)$ be a d-cluster tilting subcategory. Assume that the global dimension of $\Lambda(n,l)$ is also d.

Then the mutation graph of strongly maximal τ_d-rigid pairs of $\Lambda(n,l)$ is an extended Dynkin diagram of type $\tilde{A}$ with $2n+l-1$ vertices. In particular, it is a 2-regular graph.

Mutation graph for $n = 9$, $l = 3$, $d = 2$



Thank you!