The orbit method for the Witt algebra

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Motivation: The orbit method and the Witt algebra

The orbit method via Poisson primitive ideal

$\operatorname{Prim} \mathrm{U}(W)$ and the Dixmier map for W

Motivation: The orbit method and the Witt algebra

Lie algebra and universal enveloping algebra

- All vector spaces and Lie algebras are over $\mathbb{C}$.
- Let $\mathfrak{g}$ be a Lie algebra, $\dim \mathfrak{g}$ is finite or countable with ordered basis (e_i).
- The universal enveloping algebra $\mathrm{U}(\mathfrak{g})$

$$\mathrm{U}(\mathfrak{g}) \cong \mathbb{C}\langle e_i \rangle / (e_i e_j - e_j e_i - [e_i, e_j]).$$

Representation theory and primitive spectrum $\operatorname{Prim} \mathsf{U}(\mathfrak{g})$

- Question: Understand $\mathrm{Rep}_{\mathfrak{g}} \cong \mathrm{Mod}_{\mathsf{U}(\mathfrak{g})}$ (Irreducible representations correspond to simple modules)
- In general, difficult to understand/classify simple $\mathsf{U}(\mathfrak{g})$-modules.
- A two-sided ideal Q of $\mathsf{U}(\mathfrak{g})$ is called primitive if Q is the annihilator of a simple module M over $\mathsf{U}(\mathfrak{g}) \Leftrightarrow QM = 0$
- Refined question: Understand the primitive spectrum $\operatorname{Prim} \mathsf{U}(\mathfrak{g}) = \{\text{primitive ideals } Q \triangleleft \mathsf{U}(\mathfrak{g})\}$ instead.
- $\operatorname{Prim} \mathsf{U}(\mathfrak{g})$ is still hard to understand, so we seek some kinds of correspondence.

The orbit method and Dixmier map

- Let G be the adjoint group of $\mathfrak{g}$ acting on $\mathfrak{g}^*$ by coadjoint action. We denote the space of orbits by $\mathfrak{g}^*/G$.

Theorem (Conze, Dixmier, Duflo, Mathieu, Rentschler)

If $\mathfrak{g}$ is a finite dimensional solvable Lie algebra, then there exists a homeomorphism between

$$\mathrm{Dx} : \mathrm{Prim}\,\mathrm{U}(\mathfrak{g}) \xrightarrow[\mathrm{Dx}]{\sim} \mathfrak{g}^*/G.$$

The Witt and Virasoro algebra

- The Witt algebra $W = \mathbb{C}[t, t^{-1}]\partial$ is the Lie algebra of derivations of $\mathbb{C}[t, t^{-1}]$, where $\partial = \frac{d}{dt}$, with Lie bracket

$$[f\partial, g\partial] = (fg' - f'g)\partial.$$

- W has a nice countable basis $(e_i = t^{i+1}\partial | i \in \mathbb{Z})$, and

$$[e_i, e_j] = (j - i)e_{i+j}.$$

- Its one-dimensional central extension is the Virasoro Lie algebra *Vir*, which is important in physics, conformal field theory and vertex algebras.
- Problem: W and *Vir* do not have an adjoint group. Need a slightly different approach.

The orbit method via Poisson primitive ideal

The symmetric algebra $\mathrm{S}(\mathfrak{g})$

- A Poisson algebra A is a commutative algebra with a Poisson bracket (that is a Lie bracket) such that

$$\{xy, z\} = \{x, z\}y + \{y, z\}x.$$

- The symmetric algebra $\mathrm{S}(\mathfrak{g}) \cong \mathbb{C}[e_i]$ is a Poisson algebra with $\{x, y\} = [x, y]$ $(x, y \in \mathfrak{g})$. $\mathrm{S}(\mathfrak{g})$ is the associated graded of $\mathrm{U}(\mathfrak{g})$.

Poisson primitive spectrum of $\mathsf{S}(\mathfrak{g})$

- $\mathsf{S}(\mathfrak{g})$ satisfies (generalized) Nullstellensatz

$$\chi \in \mathfrak{g}^* \leftrightarrow \text{ maximal ideal } \mathfrak{m}_\chi = \ker(ev_\chi : \mathsf{S}(\mathfrak{g}) \to \mathbb{C}) \triangleleft \mathsf{S}(\mathfrak{g}).$$

- $I \triangleleft \mathsf{S}(\mathfrak{g})$ is a Poisson ideal if $\{I, \mathsf{S}(\mathfrak{g})\} \subseteq I$
- A Poisson primitive ideal $P(\chi)$ of $\mathsf{S}(\mathfrak{g})$ is the maximal Poisson ideal contained in $\mathfrak{m}_\chi$.
- The Poisson primitive spectrum of $\mathsf{S}(\mathfrak{g})$ $\mathrm{P.\,Prim}\,\mathsf{S}(\mathfrak{g}) = \{P(\chi) | \chi \in \mathfrak{g}^*\}$.
- We have a canonical map

$$\mathfrak{g}^* \to \mathrm{P.\,Prim}\,\mathsf{S}(\mathfrak{g}); \quad \chi \mapsto P(\chi).$$

- The set $\mathbb{O}(\chi) = \{\eta \in \mathfrak{g}^* | P(\chi) = P(\eta)\}$ is called the pseudo-orbit of χ. When $\mathfrak{g}$ is finite-dimensional with adjoint group G, then $\mathbb{O}(\chi) = G \cdot \chi$.

The orbit method via Poisson primitive ideal

Theorem ([Goodearl, 2010])

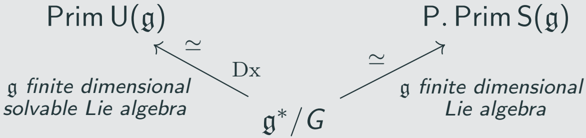
$\operatorname{Prim} \mathrm{U}(\mathfrak{g})$ $\qquad\qquad$ $\mathrm{P.Prim}\, \mathrm{S}(\mathfrak{g})$

$\mathfrak{g}^*/G \xrightarrow[\simeq]{\mathrm{Dx}} \operatorname{Prim} \mathrm{U}(\mathfrak{g})$

$\mathfrak{g}$ finite dimensional solvable Lie algebra

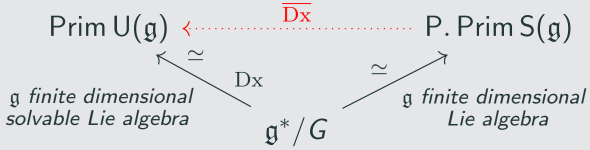
The orbit method via Poisson primitive ideal

Theorem ([Goodearl, 2010])



The orbit method via Poisson primitive ideal

Theorem ([Goodearl, 2010])



- The algebraic group G is absent from homeomorphism $\overline{\mathrm{Dx}} : \mathrm{P.} \operatorname{Prim} \mathrm{S}(\mathfrak{g}) \to \operatorname{Prim} \mathrm{U}(\mathfrak{g})$ (when $\mathfrak{g}$ is finite-dimensional solvable Lie algebra), thus can be applied in a more general setting (e.g. quantum groups).
- Goal: Construct the map $\overline{\mathrm{Dx}} : \mathrm{P.} \operatorname{Prim} \mathrm{S}(W) \to \operatorname{Prim} \mathrm{U}(W)$ for the countable dimensional Witt algebra.

Poisson primitve spectrum of $\mathsf{S}(W)$

- Goal: Study $\operatorname{Prim} \mathsf{U}(W)$ and construct $\overline{\mathrm{Dx}} : \mathrm{P.}\operatorname{Prim} \mathsf{S}(W) \to \operatorname{Prim} \mathsf{U}(W)$.
- $\mathrm{P.}\operatorname{Prim} \mathsf{S}(W)$ is understood in [Petukhov and Sierra, 2023]
- A one-point local function $\chi_{x;\alpha_0,\ldots,\alpha_n}$ on W:

$$\chi_{x;\alpha_0,\ldots,\alpha_n} : W \to \mathbb{C}; \quad f\partial \mapsto \alpha_0 f(x) + \cdots + \alpha_n f^{(n)}(x),$$

  where $x, \alpha_n \neq 0$. n is called the order of χ.
- A local function $\chi = \chi_1 + \cdots + \chi_\ell$ on W is a finite-sum of one-point local function χ_i, where $x_i \neq x_j$.
- [Petukhov and Sierra, 2023] $P(\chi) \neq 0$ iff χ is a local function on W. They also completely characterize $\mathbb{O}(\chi)$ for $\chi \in W^*$.

$\operatorname{Prim} \mathrm{U}(W)$ and the Dixmier map for W

Local representation of W from orbit method

The orbit method gives us a recipe to construct representations of W from local functions on W.

- Let $\chi = \chi_1 + \dots \chi_\ell$ be a local function on W, where χ_i is of order n_i at x_i.
- A polarization of χ is subalgebra $\mathfrak{p}$ of W such that $\chi\big|_{[\mathfrak{p},\mathfrak{p}]} = 0$ and has maximal dimension.

Lemma (P.)

A polarization for χ is $\mathfrak{p} = (t - x_1)^{m_1+1} \dots (t - x_\ell)^{m_\ell+1} W$, where $m_i = \left\lfloor \frac{n_i}{2} \right\rfloor$.

- So χ is a one-dimensional representation of $\mathfrak{p}$, which we write $\mathbb{C}_\chi$. The induced module

$$M_\chi = \mathrm{U}(W) \otimes_{\mathrm{U}(\mathfrak{p})} \mathbb{C}_\chi$$

is called a local representation of W.

Dixmier map for the Witt algebra

- Let $Q(\chi) = \mathrm{Ann}_{\mathrm{U}(W)} M_\chi$.

Theorem (P.)

Let $\chi = \chi_1 + \cdots + \chi_\ell$ be a local function on W. Then M_χ is an irreducible representation of W iff $n_i > 0$ for all i.

However, all $Q(\chi)$ are (completely prime) primitive ideals of $\mathrm{U}(W)$ (no assumption on n_i). Thus, we now have a well-defined map

$$W^* \to \mathrm{Prim}\,\mathrm{U}(W), \quad \begin{aligned} &\chi \textit{ local} \mapsto Q(\chi), \\ &\chi \textit{ not local} \mapsto 0. \end{aligned}$$

Dixmier map for the Witt algebra

Thus we have a candidate for the Dixmier map

$$\begin{array}{ccc} W* & \longrightarrow & \mathrm{P.\,Prim}(\mathrm{S}(W)) \\ & \searrow & \\ & & \mathrm{Prim}(\mathrm{U}(W)). \end{array} \qquad \begin{array}{ccc} \chi \text{ local} & \longmapsto & P(\chi) \\ & \searrow & \\ & & Q(\chi). \end{array}$$

Dixmier map for the Witt algebra

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$$
\begin{array}{ccc}
W* & \longrightarrow & \mathsf{P.\,Prim}(\mathsf{S}(W)) \\
 & \searrow & \Big\downarrow{\scriptstyle \overline{\mathrm{Dx}}} \\
 & & \mathsf{Prim}(\mathsf{U}(W)).
\end{array}
\qquad
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Dixmier map for the Witt algebra

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 & & Q(\chi).
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$$

Theorem (P.)

The Dixmier map $\overline{\mathrm{Dx}}$ above is well-defined, i.e.

$$\textit{if } P(\chi) = P(\eta), \textit{ then } Q(\chi) = Q(\eta).$$

Proof idea - Use solvable finite-dimensional algebra

- Let $\mathfrak{g}_n = \mathbb{C}\{e_0, ..., e_{n-1}\}$ be a finite-dimensional solvable subquotient W with $[e_i, e_j] = (j-i)e_{i+j}$. Let $\mathbb{C}[t, t^{-1}, \partial]$ be the localized Weyl algebra with $\partial t - t\partial = 1$.
- We consider one-point local function $\chi = \chi_{x;\alpha_0,...,\alpha_n} \in W^*$.

Theorem (P.)

There exists a graded ring homomorphism

$$\Psi_n : \mathrm{U}(W) \to T_n = \mathbb{C}[t, t^{-1}, \partial] \otimes_{\mathbb{C}} \mathrm{U}(\mathfrak{g}_n),$$

and a T_n-module L_χ such that $M_\chi \cong \mathrm{res}_{\Psi_n} L_\chi$.

- $Q(\chi) = \Psi_n^{-1}(\mathrm{Ann}_{T_n} L_\chi)$. Then we use the Dixmier map on the solvable Lie algebra $\mathfrak{g}_n$.

Questions and Future Work

We have shown that the map $\overline{\mathrm{Dx}} : \mathsf{P.\,Prim\,S}(W) \to \mathsf{Prim\,U}(W)$ is well-defined and it is indeed an instance of orbit method.

- We know that $\overline{\mathrm{Dx}}$ is not injective. But how non-injective is it, i.e. all instances where injectivity fails?
- Is $\overline{\mathrm{Dx}}$ a continuous and open map?
- Is $\overline{\mathrm{Dx}}$ surjective?
- For $\chi, \eta \in W^*$, when is $P(\chi) \subsetneq P(\eta)$, that is, what is the "closure" of $\mathbb{O}(\chi)$? When is $Q(\chi) \subsetneq Q(\eta)$?

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