# Derived equivalences for derived discrete algebras of infinite global dimension

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$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\omega_X \circ F(h)} \Sigma' F(X)$$

    is exact in $\mathcal{T}'$.
- $\eta\colon (F, \omega) \to (F', \omega')$ is a **natural transformation** between triangle functors if $\eta$ is a natural transformation between $F$ and $F'$ st. $\omega' \circ \eta\Sigma = \Sigma'\eta \circ \omega$.

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**Open question (Rickard '91)**

Is any derived equivalence standard?

**Affirmative answer for algebras:**

- hereditary (Miyachi, Yekutieli '01),
- (anti-)Fano (Minamoto '12),
- triangular (Chen '16),
- **derived discrete of finite global dimension** (Chen, Zhang '19).

# Derived discrete algebras

## Definition (Vossieck '01)

A finite dimensional $\mathbb{K}$-algebra $A$ is **derived discrete** if for any $\mathbf{n} = (n_i)_{i\in\mathbb{Z}}$, where $n_i \in \mathbb{N}_{\geq 0}$, there are only finitely many isoclasses of indecomposables in $\mathbf{D}^b(\operatorname{mod} A)$ of cohomology dimension vector $\mathbf{n}$.

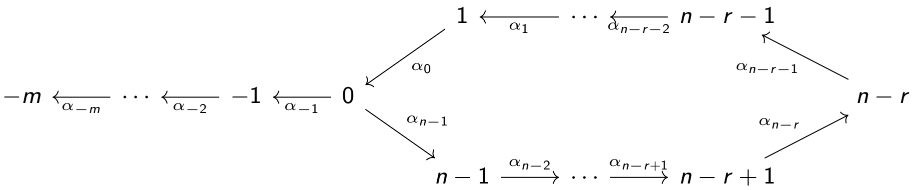
## Example

The algebra of dual numbers given as a path algebra of the quiver:

$$0 \circlearrowleft \alpha_0$$

where $\alpha_0^2 = 0$

Let $A$ be a finite dimensional algebra which is not piecewise hereditary of Dynkin type. Then $A$ is derived discrete iff $\mathbf{D}^b(\operatorname{mod} A) \cong \mathbf{D}^b(\operatorname{mod} \Lambda(r,n,m))$, where $1 \leq r \leq n$, $m \geq 0$ and $\Lambda(r,n,m) = \mathbb{K}\Delta(r,n,m)/I(r,n,m)$ is the path algebra of the quiver $\Delta(r,n,m)$:




bounded by $\alpha_{n-1}\alpha_0$, $\alpha_{n-2}\alpha_{n-1}$, ..., $\alpha_{n-r}\alpha_{n-r+1}$.
Moreover, gldim $A = \infty$ iff $r = n$.
**Ref**: Vossieck '01 (Thm 2.1) and Bobiński-Geiss-Skowroński '04 (Thm A)

# Useful criterion

- Let $\mathcal{A}$ be a category. A triangle functor $(F,\omega)\colon \mathbf{K}^b(\mathcal{A}) \to \mathbf{K}^b(\mathcal{A})$ is a **pseudo-identity** if:
  - $F(X) = X$ for all objects $X \in \mathcal{A}$,
  - $F|_{\Sigma^n(\mathcal{A})} = \mathrm{Id}_{\Sigma^n(\mathcal{A})}$.
- Note: any pseudo-identity is an autoequivalence.

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**Theorem (Chen, Ye '18)**

Let $A$ be a finite dimensional algebra. Suppose that any pseudo-identity $(F, \omega)$ on $\mathbf{K}^b(\mathrm{proj}\, A)$ is isomorphic to $(\mathrm{Id}_{\mathbf{K}^b(\mathrm{proj}\, A)}, \mathrm{Id}_\Sigma)$. Then for any finite dimensional algebra $B$ any derived equivalence $\mathbf{D}^b(\mathrm{mod}\, A) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{mod}\, B)$ is standard.

# Strategy of the proof

## Main result

Any derived equivalence between derived discrete algebras of infinite global dimension is standard.

## Remark

A partial result has been obtained in [Chen, Ye, '18, Section 7] for the algebras of type $\Lambda(n, n, 0)$.

# Strategy of the proof

## Main result

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## Remark

A partial result has been obtained in [Chen, Ye, '18, Section 7] for the algebras of type $\Lambda(n, n, 0)$.

We will start by describing a structure of ind $\mathbf{K}^b(\operatorname{proj}\Lambda)$ for $\Lambda = \Lambda(n, n, m)$. Then:

0. pick an arbitrary pseudo-identity $(F, \omega)$ on $\mathbf{K}^b(\operatorname{proj}\Lambda)$,
1. construct an isomorphism $\eta\colon F|_{\operatorname{ind}\mathbf{K}^b(\operatorname{proj}\Lambda)} \to \operatorname{Id}_{\operatorname{ind}\mathbf{K}^b(\operatorname{proj}\Lambda)}$ (which extends to an isomorphism $\eta\colon F \to \operatorname{Id}_{\mathbf{K}^b(\operatorname{proj}\Lambda)}$),
2. then, by [Chen, Ye '18, Lemma 2.1] we obtain $(F, \omega) \xrightarrow{\sim} (\operatorname{Id}, \omega')$ for some $\omega'$,
3. depending on $n, m$, either $\omega' = \operatorname{Id}_\Sigma$ or $(\operatorname{Id}, \omega') \cong (\operatorname{Id}, \operatorname{Id}_\Sigma)$.

$\operatorname{ind} \mathbf{K}^b(\operatorname{proj} \Lambda) \cong \mathbb{K}\Gamma/\mathfrak{I}$ where $\Gamma = (\Gamma_0, \Gamma_1)$ and:

$$\Gamma_0 = \{(i, a, b) \mid i \in \{0, \dots, n-1\}, a \leq b + \delta_{i,0} \cdot m\}.$$

For every $v = (i, a, b) \in \Gamma_0$ we define two sets:

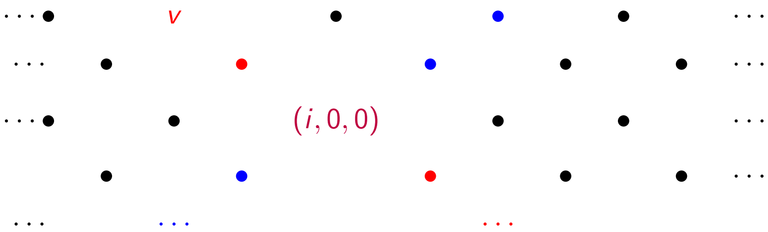
$$\mathcal{I}_v = \{(i, x, y) \mid x = a, \dots, b + \delta_{i,0} \cdot m,\ y \geq b\},$$

$$\mathcal{X}_v = \{(i+1, x, y) \mid x \leq a + \delta_{i,n-1} \cdot m,\ y = a, \dots, b + \delta_{i,0} \cdot m\}.$$

- for $u \in \mathcal{I}_v$ st. $u \neq v$ we have $f_{v,u} \colon v \to u$ in $\Gamma_1$,
- for $u \in \mathcal{X}_v$ we have $e_{v,u} \colon v \to u$ in $\Gamma_1$,
- note: $i+1$ is taken modulo $n$.
- **Ref**: Bobiński '11

# Example $\mathcal{I}_v$ and $\mathcal{X}_v$ sets

$n = 1, m = 2, \ v = (0, 0, -m)$, we have sets $\mathcal{I}_v$ and $\mathcal{X}_v$, intersection $\mathcal{I}_v \cap \mathcal{X}_v$ is in purple.



# Structure of the homotopy category

The arrows are subject to the following relations for every $v \in \Gamma_0$:

1. For $u \in \mathcal{I}_v, w \in \mathcal{I}_u, v \neq u, w \neq u$:

$$f_{u,w} \circ f_{v,u} = \begin{cases} f_{v,w} & w \in \mathcal{I}_v, \\ 0 & \text{otherwise.} \end{cases}$$

2. For $u \in \mathcal{I}_v, w \in \mathcal{X}_u, v \neq u$:

$$e_{u,w} \circ f_{v,u} = \begin{cases} e_{v,w} & w \in \mathcal{X}_v, \\ 0 & \text{otherwise.} \end{cases}$$

3. For $u \in \mathcal{X}_v, w \in \mathcal{I}_u$:

$$f_{u,w} \circ e_{v,u} = \begin{cases} e_{v,w} & w \in \mathcal{X}_v, \\ 0 & \text{otherwise.} \end{cases}$$

4. For $u \in \mathcal{X}_v, w \in \mathcal{X}_u$: $e_{u,w} \circ e_{v,u} = 0$.

- If $v = u$ then we define $f_{v,v}$ to be the identity morphism $\mathrm{id}_v$.
- We write $e'_{v,u} = \begin{cases} e_{v,u} & u \in \mathcal{X}_v, \\ 0 & \text{otherwise.} \end{cases}$
- We have the following definitions of the suspension functor $\Sigma$ and the AR-translation $\tau$:
  - $\Sigma(i, a, b) = (i + 1, 1 + \delta_{i,n-1} \cdot m, 1 + \delta_{i,0} \cdot m)$
  - $\tau(i, a, b) = (i, a - 1, b - 1)$.

# Structure of the homotopy category

**Lemma**

Let $v = (i, a, b) \in \Gamma_0$. Then:

1. $f \in \mathrm{End}(v)$ is an automorphism iff $f = \lambda f_{v,v} + \mu e'_{v,v}$ for $\lambda \neq 0$.
2. Suppose we have $f : v \to u$. Then $f$ is irreducible iff $u \in \{(i, a+1, b), (i, a, b+1)\}$ and $f = \lambda f_{v,u} + \mu e'_{v,u}$ for $\lambda \neq 0$.

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**Lemma**

We have identification of projective modules up to $\Sigma$ and AR-translation: $P_{-j}$ with $(0, 0, -j)$ for $j = 0, \ldots, m$ and $P_k$ with $(k, 0, 0)$ for $k = 1, \ldots, n-1$ (compare: Bobiński, '10, Corollary 6.3).

# Proof – part 1

We construct $\phi_v \in \mathrm{Aut}(v)$ for every $v \in \Gamma_0$, st. $F'(f) = f$ for every $f$ where $F'(f) := \phi_u \circ F(f) \circ \phi_v^{-1}$ for $f \colon v \to u$.

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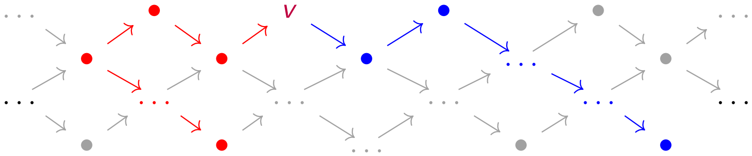
This will be done inductively. We put $\phi_v = \mathrm{id}_v$ for $v = (i, 0, -\delta_{i,0} \cdot m)$.
For the rest, we construct *canonical walks to and from* $v$ for every connected component of the AR-quiver:



# Proof – part 1

Fix $v = (i, a, b) \in \Gamma_0$ and let $f : v \to u$ be an irreducible morphism.

1. If $u = (i, a, b+1) \in \Gamma_0$, then there exists $\phi \in \mathrm{Aut}(u)$, st. $\phi \circ f = f_{v,u}$.
2. If $u = (i, a+1, b+1)$ for $b = a - \delta_{i,0} \cdot m$ such that $f \circ f_{w,v} = 0$ for $w = (i, a, b)$, then there exists $\phi \in \mathrm{Aut}(u)$ st. $\phi \circ f = f_{v,u}$.

Dual statements for "backward" canonical walks also hold. For the proof of the above observation, let $f = \lambda f_{v,u} + \mu e'_{v,u}, \lambda \neq 0$. Then:

1. Put $\phi := \lambda^{-1} f_{u,u} - \lambda^{-2} \mu e'_{u,u}$.
2. Three subcases:
   - $e'_{u,u} = 0, e'_{v,u} = 0$. Proceed as in (1).
   - $e'_{u,u} = e_{u,u}, e'_{v,u} = e_{v,u}$. Proceed as in (1).
   - $e'_{u,u} = 0, e'_{v,u} = e_{v,u}$. This happens iff $n = 1, m = 1$. Note $u \notin \mathcal{I}_w$ but $u \in \mathcal{X}_w$. Hence $0 = f \circ f_{w,v} = \mu e_{w,u} \implies \mu = 0 \implies \phi := \lambda^{-1} f_{u,u}$.

By applying the above observations we inductively construct $\phi_u$ for every $u \in \Gamma_0$ such that $F'(f_{v,u})(= \phi_u \circ F(f_{v,u}) \circ \phi_v^{-1}) = f_{v,u}$.

If $n = 1$, then for brevity we will use $(a, b) := (0, a, b)$ for indecomposables.

# Proof – part 2

In this step we will show that $F'(e_{v,u}) = e_{v,u}$ for every morphism $e_{v,u} \colon v \to u$.

## General method

Find decomposition of morphisms of one of the forms:

- $e_{w_1,w_4} = f_{w_3,w_4} \circ e_{w_2,w_3} \circ f_{w_1,w_2}$ where either $e_{v,u} = e_{w_1,w_4}$ or $e_{v,u} = e_{w_2,w_3}$, or
- $f_{w_2,w_3} \circ e_{w_1,w_2} = e_{w,w_3} \circ f_{w_1,w}$ where either $e_{v,u} = e_{w_1,w_2}$ or $e_{v,u} = e_{w,w_3}$,

where, by induction, $F'$ acts trivially on all the morphisms different from $e_{v,u}$. Obtain the result by applying $F'$ to the both sides of the equation.

# Proof – part 3

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## Lemma

Let $(\mathrm{Id}_{\mathbf{K}^b(\mathrm{proj}\,\Lambda)}, \omega') \colon \mathbf{K}^b(\mathrm{proj}\,\Lambda) \to \mathbf{K}^b(\mathrm{proj}\,\Lambda)$. Then for every $v \in \mathrm{ind}(\mathbf{K}^b(\mathrm{proj}\,\Lambda))$ we get $\omega'_v = f_{\Sigma v, \Sigma v} + \mu e'_{\Sigma v, \Sigma v}$ where $\mu \in \mathbb{K}$. In particular if $n > 1$, then $\omega' = \mathrm{id}_\Sigma$.

## Idea of proof

$v = (i, a, b)$, $v = (i, a, b+1)$, $w = (i, b + \delta_{i,0} \cdot m + 1, b+1)$. There exists exact triangle: $v \xrightarrow{f_{v,u}} u \xrightarrow{f_{u,w}} w \xrightarrow{\lambda e_{w,\Sigma v}} \Sigma v$ (see Bobiński, Schmude '20, Prop. 2.2). Apply $(\mathrm{Id}_{\mathbf{K}^b(\mathrm{proj}\,\Lambda)}, \omega)$ and use (TR3) axiom.

# Proof – part 3

It suffices to consider $n = 1$.

- If $m > 0$, then in every ray of the AR-quiver we have $v = (a, b)$, st. $\dim_{\mathbb{K}}(\mathrm{End}(v)) = 1$, hence $\omega'_v = \mathrm{id}_{\Sigma v}$. If $f : v \to u$ for $u = (a, b + 1)$, then by functoriality $\omega'_u \circ \Sigma(f) = \Sigma(f)$. But since $\mathrm{Hom}(v, u) \cong \mathrm{End}(u)$, as left $\mathrm{End}(u)$-modules then $\omega'_u = \mathrm{id}_{\Sigma u}$. Proceed by induction on $b$.

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- If $m = 0$, then $(\mathrm{Id}_{\mathbf{K}^b(\mathrm{proj}\,\Lambda)}, \omega') \cong (\mathrm{Id}_{\mathbf{K}^b(\mathrm{proj}\,\Lambda)}, \mathrm{Id}_{\Sigma})$. Isomorphism is constructed iteratively. For $v = (0, b)$ $b \geq 0$ put $\eta_v := \mathrm{id}_v$. Let $s \geq 0$. For $v = (s, b) \in \Gamma_0$ put:

$$\eta_v = \Sigma^{s-1}(\omega_{(0,b-s)}) \circ \Sigma^{s-2}(\omega_{(1,b-s+1)}) \circ \ldots \circ \omega_{(s-1,b-1)}$$

For $v = (-s, b) \in \Gamma_0$ put:

$$\eta_v = \Sigma^{-s}(\omega^{-1}_{(-1,b+s-1)}) \circ \Sigma^{-s+1}(\omega^{-1}_{(-2,b+s-2)}) \circ \ldots \circ \Sigma^{-1}(\omega^{-1}_{(-s,b)})$$