The Module Structure of a Group Action on a Ring

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Context

Notation: k a field, of finite characteristic p to be interesting, occasionally assumed to be algebraically closed to avoid trivialities.

S a noetherian k-algebra graded by $\mathbb{N}$, e.g. a polynomial ring $S = k[x_1, \ldots, x_n]$.

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G a finite group, which acts on S preserving the grading.

Any module over S or some noetherian graded sub-algebra will be assumed to be graded and finite, meaning finitely generated, not finite cardinality.

Context contd.

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In the case of a polynomial ring we are just taking a finite module for a group (or a Hopf algebra) and trying to describe the symmetric powers.

Motivation

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Maybe restrict to a finite subgroup.

We would also like some information about the multiplication.

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Galois Module Theory: L a number field, G acts, $K = L^G$.

Regard $\mathcal{O}_L$ as an $\mathcal{O}_K G$-module, or perhaps just as a $\mathbb{Z}G$-module.

$\mathcal{O}_L$ has a free submodule of finite index. It is locally free if L/K is tamely ramified.

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From yesterday:

G acts on a curve C, hence on $H^0(C, \Omega_{C/k}^{\otimes m})$.

Automorphisms of $H^*(BT^n; \mathbb{F}_p)$.

Example

Field k of characteristic 3, $U = U_3(\mathbb{F}_3) = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} : * \in \mathbb{F}_3 \right\}$

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Ring $S = S(V^*) = k[V]$

Invariants:

$$d_x = x^* \qquad \text{degree 1}$$

$$d_y = \prod_{\lambda \in \mathbb{F}_3} (y^* + \lambda x^*) = y^{*3} - y^* x^{*2} \qquad \text{degree 3}$$

$$d_z = \prod_{\lambda, \mu \in \mathbb{F}_3} (z^* + \lambda y^* + \mu x^*) = z^{*9} + \cdots \qquad \text{degree 9}$$

In fact $S^U = k[d_x, d_y, d_z]$.

Example contd.

U acts on the dual space V as $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} : * \in \mathbb{F}_3 \right\}$, basis x, y, z.

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U acts on the dual space V as $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} : * \in \mathbb{F}_3 \right\}$, basis x, y, z.

U fixes z.

$U_x = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix} : * \in \mathbb{F}_3 \right\}$ fixes $\langle y, z \rangle$ pointwise.

$U_y = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix} : * \in \mathbb{F}_3 \right\}$ fixes $\langle x, z \rangle$ pointwise.

$U_3(\mathbb{F}_3)$

degree	dimensions of indecomposable summands										
0	1										
1	3										
2	6										
3	10										
4	15										
5	21										
6	$3'$	9	16								
7	9	$9'$	18								
8	9	18	$18'$								
9	1	$3'$	9	18	24						
10	3	9	$9'$	18	27						
11	6	9	18	$18'$	27						
12	$3'$	9	10	18	24	27					
13	9	$9'$	15	18	27	27					
14	9	18	$18'$	21	27	27					
15	$3'$	$3'$	9	9	16	18	24	27	27		
16	9	9	$9'$	$9'$	18	18	27	27	27		
17	9	9	18	18	$18'$	$18'$	27	27	27		
18	1	$3'$	$3'$	9	9	18	18	24	24	27	27	27
19	3	9	9	$9'$	$9'$	18	18	27	27	27	27	27
20	6	9	9	18	18	$18'$	$18'$	27	27	27	27	27

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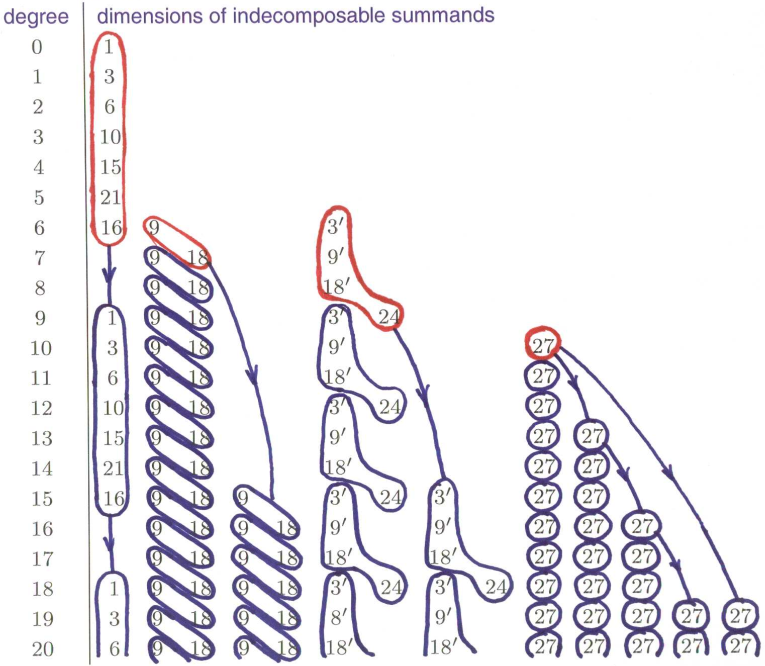
degree
dimensions of indecomposable summands
0 1
1 3
2 6
3 10
4 15
5 21
6 16 9 3′
7 9 18 9′
8 9 18 18′
9 1 9 18 3′ 24
10 3 9 18 9′ 27
11 6 9 18 18′ 27
12 10 9 18 3′ 24 27
13 15 9 18 9′ 27 27
14 21 9 18 18′ 27 27
15 16 9 18 9 3′ 24 3′ 27 27
16 9 18 9 18 9′ 9′ 27 27 27
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19 3 9 18 9 18 8′ 9′ 27 27 27 27 27
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Spot the pattern.

Commutative Algebra

Dimension: A finite R-module has **dimension** d if it is **finite** over some polynomial subalgebra $k[a_1, \ldots, a_d] \leq R$, but not for any **smaller** d.

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Later revised to: Life is worth living. Period.

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Notice that $k[V, M] \cong \mathrm{Hom}_{kG}(M^*, k[V])$ and $k[V]^G \cong \mathrm{Hom}_{kG}(k, k[V])$, so we mostly consider functors related to $\mathrm{Hom}_{kG}(M, -)$.

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For example, $\widehat{\mathrm{Ext}}^i_{kG}(M, -)$ or $\mathrm{Hom}_{kG}(P_U, -)$, where U is a simple kG-module and P_U is its projective cover. The latter counts the multiplicity of U as a composition factor.

Functors Contd.

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More concretely, for a finite dimensional indecomposable kG-module M, let

$$J(M, N) = \{f \in \mathrm{Hom}_{kG}(M, N) \quad | \; f \text{ is } \textbf{not} \text{ split injective}\}$$

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Alternatively, $J(M, N)$ is the k-span of the $f \in \mathrm{Hom}_{kG}(M, N)$ that factor through an indecomposable module that is not isomorphic to M.

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$\mathrm{Hom}^{\oplus}_{kG}(M, M) = \mathrm{End}_{kG}(M)/\operatorname{rad}\mathrm{End}_{kG}(M) \cong k$ (k algebraically closed.)

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Define $S^{\oplus G} = \mathrm{Hom}^{\oplus}_{kG}(k,S)$. It is naturally a ring: the ring of trivial summands of S.

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We will consider $S^{[G]}$. It is naturally a ring and is finite as an S^G-module.

Fixed Point Sets

Given $H \leq G$, define $I_H \leq S$ to be the ideal generated by all elements of the form $(h-1)s$ for $h \in G$, $s \in S$. Let $I = \cap_{H \in \mathrm{Syl}_p(G)} I_H$.

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The ideal $S^G \cap I \leq S^G$ gives us $V^G \leq V/G$.

Fixed Point Sets Contd.

Theorem The natural homomorphisms of k-algebras

$$S^{[G]} \twoheadrightarrow S^{\oplus G} \twoheadrightarrow S^G/(S^G \cap I) \hookrightarrow (S/I)^G$$

induce universal homeomorphisms on the spectra.

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Note that $\mathrm{Spec}((S/I)^G) \cong \mathrm{Spec}(S)^P/N_G(P)$, where P is a Sylow p-subgroup.

Fixed Point Sets Contd.

Corollary When G is a p-group and V is a kG-module we have

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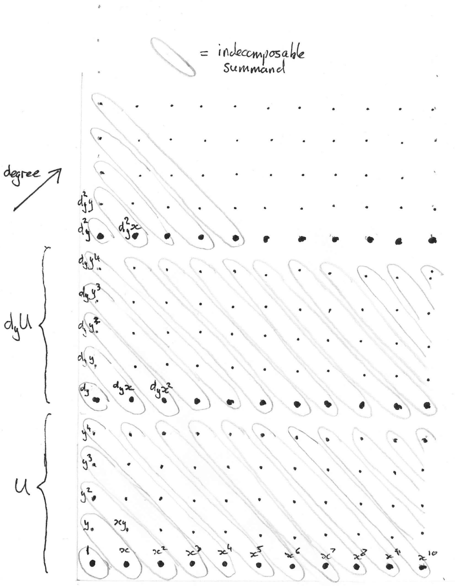
Theorem $\mathrm{Spec}(\hat{H}^0(G;S)) \cong (\mathrm{Sing}_p \, \mathrm{Spec}(S))/G$, where Sing_p means fixed by an element of order p.

Example: Cyclic Group and Two Variables



$G = C_p$ cyclic order p

$S = k[x,y]$

$d_y = y^p - x^{p-1}y$

$S^G = k[x, d_y]$

Action: $y \mapsto y + x$, $x \mapsto x$

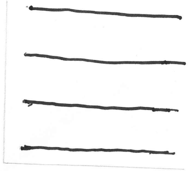
k-basis: $\{ x^i y^j d_y^k : 0 \leq j \leq p-1 \}$

$U =$ span of $\{ x^i y^j : 0 \leq j \leq p-1 \}$

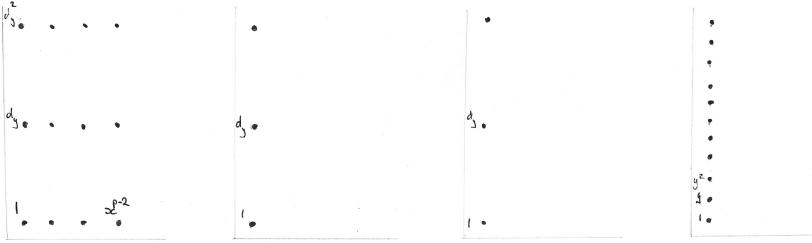
U is a $k[x]G$ - submodule of S

$S \cong k[d_y] \otimes U$ as a $k[x, d_y]G$-module

$S^G = \operatorname{soc}_{kG} S = k[x, d_y] =$



Example Contd.: Cyclic Group and Two Variables



$k[d_y] \otimes \langle 1, \ldots, x^{p-2} \rangle$ $\quad$ $k[d_y]$ $\quad$ $k[d_y]$ $\quad$ $k[y]$

$$k[V]^{[G]} \longrightarrow k[V]^{\oplus G} \longrightarrow k[V^G < V/G] \longrightarrow k[V^G < V]$$

Depth and Dimension

The previous example easily generalises.

Proposition Let G be a p-group, V a kG-module, $\dim_k V = n$, $\dim_k V^G = r$. Then there are elements $d_1, \ldots, d_n$ such that:

- $k[V^G]$ is finite over $k[d_1, \ldots, d_r]$,
- there is a finite $k[d_{r+1}, \ldots, d_n]G$-submodule $U \leq k[V]$,
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Most functors commute with $k[d_1, \ldots, d_r] \otimes_k$, so we automatically get depth $\geq r$.

Depth and Dimension

The previous example easily generalises.

Proposition Let G be a p-group, V a kG-module, $\dim_k V = n$, $\dim_k V^G = r$. Then there are elements $d_1, \ldots, d_n$ such that:

- $k[V^G]$ is finite over $k[d_1, \ldots, d_r]$,
- there is a finite $k[d_{r+1}, \ldots, d_n]G$-submodule $U \leq k[V]$,
- $S = k[d_1, \ldots, d_r] \otimes_k U$.

Most functors commute with $k[d_1, \ldots, d_r] \otimes_k$, so we automatically get depth $\geq r$.

We can leverage this.

Depth and Dimension Contd.

Let M be an indecomposable kG-module with vertex P and source U. The inertia subgroup I is the stabiliser in $N_G(P)$ of the isomorphism class of U. Let I_p denote its Sylow p-subgroup.

Depth and Dimension Contd.

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Theorem We have

$$\dim \operatorname{Hom}^{\oplus}_{kG}(M, k[V]) \leq \dim V^P.$$

If $\operatorname{Hom}^{\oplus}_{kG}(M, k[V]) \neq 0$, then

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Corollary If $I_p = P$ then $\operatorname{Hom}_{kG}^{\oplus}(M, k[V])$ is Cohen-Macaulay of dimension $\dim_k V^P$, in particular if P is a Sylow p-subgroup. The ring of trivial summands $k[V]^{\oplus G}$ is always Cohen-Macaulay of dimension V^{G_p}.

Depth and Dimension Contd.

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If the theorem doesn't hold for one functor, try another.

Theorem $\mathrm{Hom}_k(M, k[V])^{[G]}$ is Cohen Macaulay of dimension $\dim V^P$.

More on $k[V]$

Theorem For $k[V]$, only a finite number of isomorphism types of indecomposable modules occur.

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reg is the Castelnuovo-Mumford regularity.

It follows that if we know that $\operatorname{Hom}^{\oplus}_{kG}(M, k[V])$ is finite over $R = k[d_1, \ldots, d_r] \leq k[V]^G$ then $\operatorname{Hom}^{\oplus}_{kG}(M, k[V])$ has generators and relations as an R-module in degrees at most $\sum(\deg d_i - 1)$.

More on $k[V]$ contd.

More is true. If a computer can calculate the multiplicity of M as a summand of S in degrees up to $\sum(\deg d_i - 1)$ then we can deduce the multiplicity in all degrees by a simple combinatorial formula.

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In fact we can always manage $d_i = |G|$, so if we know $k[V]$ as a kG-module in degrees up to $(\dim_k V)(|G| - 1)$ then we know it in all degrees.