Categorification of Collapsing Subsurfaces

Suiqi Lu
Based on a joint work with L. Fan and Y. Qiu

Tsinghua University

ICRA 21, Shanghai
2024.08.08

Contents

Motivation

Subsurface collapsing in [Barbieri-Möller-Qiu-So 2022]:

- $\mathbf{S}_\Delta$: a decorated marked surface.
- Σ: a subsurface of $\mathbf{S}_\Delta$ (can be regarded as a decorated marked surface).
- $\overline{\mathbf{S}}_{\mathrm{w}}$: the collapsed surface (a weighted decorated marked surface).

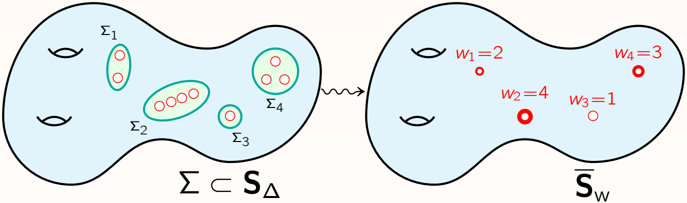


Figure: An example

Motivation

- $\mathcal{D}_3(\mathbf{S}_\Delta)$ and $\mathcal{D}_3(\Sigma)$: are Calabi-Yau-3 categories associated with $\mathbf{S}_\Delta$ and Σ respectively.
- $\mathcal{D}(\overline{\mathbf{S}}_{\mathrm{w}})$: is defined as the Verdier quotient

$$\mathcal{D}(\overline{\mathbf{S}}_{\mathrm{w}}) := \mathcal{D}_3(\mathbf{S}_\Delta)/\mathcal{D}_3(\Sigma).$$

Theorem (BMQS)

$$\mathrm{Stab}\,\mathcal{D}(\overline{\mathbf{S}}_{\mathrm{w}}) \cong \mathrm{FQuad}(\overline{\mathbf{S}}_{\mathrm{w}}).$$

Key points of our work

- We allow normal points and simple poles (among zeros) as finite singularities, i.e. the initial surface is a general wDMS and we allow any weight $\geqslant -1$.
- We deal with more general type categories instead of CY-3 ones.

Contents

wDMS

Definition

A weighted decorated marked surface (wDMS) is $\mathbf{S}_w = (\mathbf{S}, \mathbf{M}, \Delta, \mathbf{w})$, where

- $\mathbf{S}$ is a compact oriented surface, possibly with boundary $\partial\mathbf{S}$,
- $\mathbf{M} \subset \mathbf{S}$ is a finite subset of marked points,
- $\Delta \subset \mathbf{S}^\circ := \mathbf{S} \backslash \partial\mathbf{S}$ is a finite subset of decorations,
- $\mathbf{w} \colon \Delta \to \mathbb{Z}_{\geqslant -1}$ is the weight function,

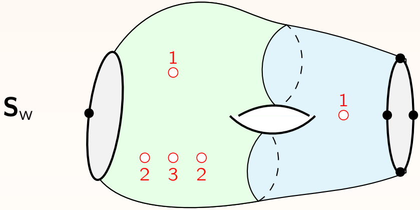
such that

1. $\mathbf{M} \cap \Delta = \varnothing$,
2. each component of $\partial\mathbf{S}$ contains at least one point of $\mathbf{M}$,
3. $\mathbf{w}$ is compatible with $\mathbf{S}$, i.e.

$$\|\mathbf{w}(z)\| - (m + 2b) = 4g - 4\,, \tag{1}$$

where $\|\mathbf{w}(z)\| = \sum_{z\in\Delta} \mathbf{w}(z)$, m is the number of marked points on $\partial\mathbf{S}$, b is the number of boundary components of $\mathbf{S}$ and g is genus of $\mathbf{S}$.

An example of wDMS



Mixed-angulations

- An arc c in $\mathbf{S}_w$ is a curve $c : [0,1] \to \mathbf{S}$ such that $c(t) \in \mathbf{S}^\circ$ for any $t \in (0,1)$.
- An open arc is an arc γ whose endpoints $\gamma(0)$ and $\gamma(1)$ are in $\mathbf{M}$.
- A closed arc is an arc η whose endpoints $\eta(0)$ and $\eta(1)$ are in Δ.
- A mixed-angulation $\mathbb{A}$ of a weighted DMS $\mathbf{S}_w$ is a finite collection of open arcs on $\mathbf{S}_w$ with no (self-)intersection between any of them in $\mathbf{S}_w^\circ := \mathbf{S}^\circ \backslash \Delta$. These arcs divide $\mathbf{S}_w$ into once-decorated polygons, where each decoration z with weight $\mathbf{w}(z)$ resides within a $(\mathbf{w}(z)+2)$-gon. We denote such a $(\mathbf{w}(z)+2)$-gon by $\mathbb{A}(z)$ and refer to it as an $\mathbb{A}$-polygon.
- We fix an initial mixed-angulation $\mathbb{A}_0$ and it induces a grading on $\mathbf{S}_w$, which is a section of $\mathbb{P}T(\mathbf{S}_w \backslash \Delta)$. Then $\mathbf{S}_w$ becomes a graded surface, and arcs have gradings and intersection indices.

Mixed-angulations

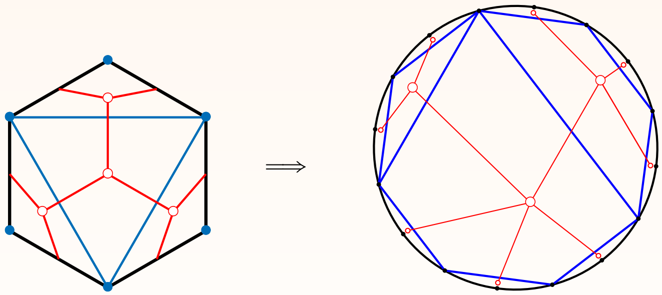


Figure: Triangulation $\Longrightarrow$ Mixed-angulation

Forward Flips of Mixed-angulations

Definition

Let $\mathbb{A}$ be a mixed-angulation of $\mathbf{S}_\mathrm{w}$. The forward flip $\mathbb{A}_\gamma^\sharp$ of $\mathbb{A}$ with respect to a graded arc $\gamma \in \mathbb{A}$ is defined as follows.

- If the endpoints of γ do not coincide, we define the new open arc $\gamma^\sharp$ by moving the endpoints of γ along the $\mathbb{A}$-gons containing γ anti-clockwisely.
- If γ is a monogon arc, we define new open arc $\gamma^\sharp$ by moving both the endpoints of γ together along the $\mathbb{A}$-gons containing γ anti-clockwisely.

We obtain a new mixed-angulation $\mathbb{A}_\gamma^\sharp$ by replacing γ with $\gamma^\sharp$. The backward flip $\mathbb{A}_\gamma^\flat$ is the inverse of a forward flip.

- The exchange graph $\mathrm{EG}(\mathbf{S}_\mathrm{w})$ associated with a weighted DMS $\mathbf{S}_\mathrm{w}$ is a directed graph. Its vertices represent mixed-angulations, and its directed edges denote forward flips between them. Fix an initial mixed-angulation $\mathbb{A}_0$ of $\mathbf{S}_\mathrm{w}$, and denote by $\mathrm{EG}^\circ(\mathbf{S}_\mathrm{w})$ the connected component of $\mathrm{EG}(\mathbf{S}_\mathrm{w})$ that includes $\mathbb{A}_0$.

An example for Case 1

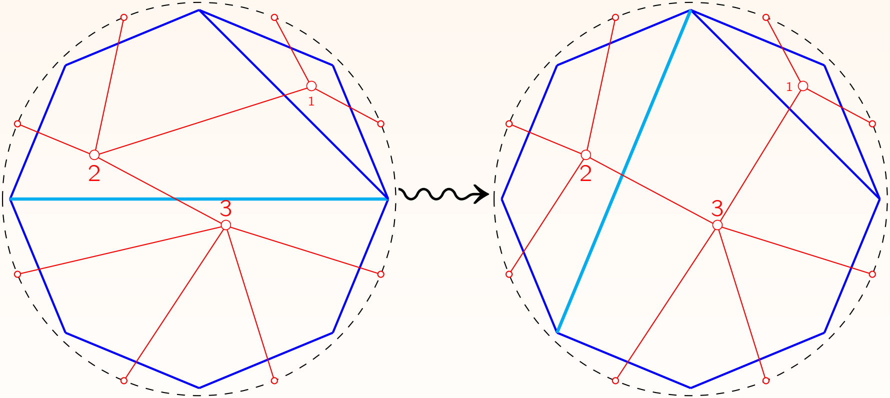


Figure: The forward flip at a usual arc.

An example for Case 2

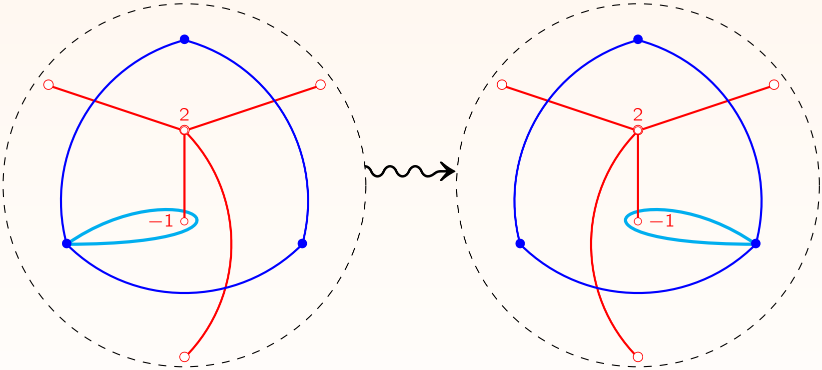


Figure: The forward flip at a monogon arc.

Contents

Collapsing subsurfaces

Definition

- $\mathbf{S}_\mathrm{w}$: *a wDMS.*
- Σ: *a subsurface of* $\mathbf{S}_\mathrm{w}$ *with connected components* Σ_i.
- c_{ij}: *(simple closed) curves such that the union* $\cup_j c_{ij}$ *forms* $\partial\Sigma_i$.
- *An assignment of integers* $\kappa_{ij} \geqslant 1$ *to each curve* c_{ij} *is called an* *enhancement* *if for each* i*, such that*

$$\sum_{z\in\Delta\subset\Sigma_i} \mathbf{w}(z) - \sum_j (\kappa_{ij}+2) = 4g(\Sigma_i) - 4.$$

- *The* *collapse* *of* Σ *in* $\mathbf{S}_\mathrm{w}$*: the weighted DMS* $\overline{\mathbf{S}}_{\overline{\mathrm{w}}}$ *obtained by filling each boundary* c_{ij} *in* $\mathbf{S}_\mathrm{w}\backslash\Sigma$ *by a disc with one decorated point that carries the weight* $w_{ij} = \kappa_{ij} - 2$.

- We regard Σ as a wDMS with κ_{ij} marked points on each boundary component.
- Note that if c_{ij} is a boundary component of $\mathbf{S}_\mathrm{w}$, then κ_{ij} coincides with the number of marked points of $\mathbf{S}_\mathrm{w}$ on c_{ij}.

An Example

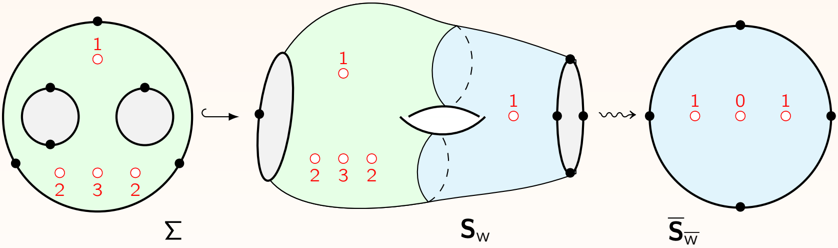


Figure: A collapse with $\kappa_{11} = 3, \kappa_{12} = 2$.

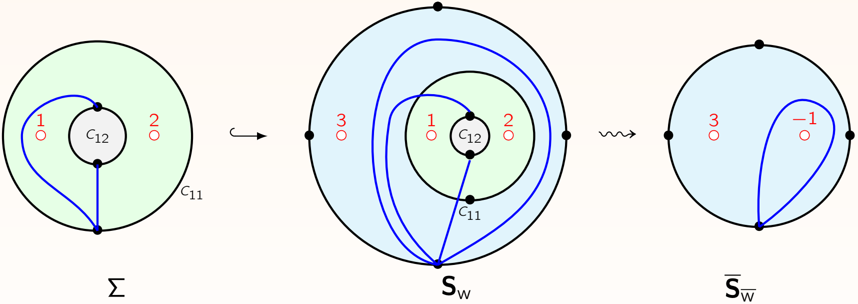
Refinements

Definition

Let $\mathbb{A}$ be a mixed-angulation of $\mathbf{S}_{\mathrm{w}}$ and $\overline{\mathbb{A}}$ be a mixed-angulation of the collapsed surface $\overline{\mathbf{S}}_{\overline{\mathrm{w}}}$. We say that $\mathbb{A}$ is a refinement of $\overline{\mathbb{A}}$ if $\overline{\mathbb{A}}$ is isotopic to a subset of $\mathbb{A}$.

- Recall that $\mathrm{EG}^{\circ}(\mathbf{S}_{\mathrm{w}})$ is the connected component of $\mathrm{EG}(\mathbf{S}_{\mathrm{w}})$ containing the initial mixed-angulation $\mathbb{A}_0$. The principal part $\mathrm{EG}^{\bullet}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}})$ of $\mathrm{EG}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}})$ is defined as the full subgraph of $\mathrm{EG}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}})$ consisting of the mixed-angulations which admit a refinement lying in $\mathrm{EG}^{\circ}(\mathbf{S}_{\mathrm{w}})$.

An Example of Refinements



Properties of Refinements

Lemma

Let $\mathbb{A}$ be a refinement of a mixed-angulation $\overline{\mathbb{A}}$ in $\mathrm{EG}^{\bullet}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}})$. Then each flip of $\mathbb{A}$ with respect to an open arc which is not under the image of collapsing is a refinement of $\overline{\mathbb{A}}$.

Lemma (generalized from BMQS)

Let $\overline{\mathbb{A}}$ be a mixed-angulation in $\mathrm{EG}^{\bullet}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}})$ and $\overline{\mathbb{A}} \xrightarrow{\overline{\gamma}} \overline{\mathbb{A}}^{\sharp}_{\overline{\gamma}}$ be a forward flip in $\mathrm{EG}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}})$. If $\overline{\gamma}$ does not form a monogon, it can be lifted to a forward flip $\mathbb{A} \xrightarrow{\gamma} \mathbb{A}^{\sharp}_{\gamma}$ in $\mathrm{EG}^{\circ}(\mathbf{S}_{\mathrm{w}})$. If $\overline{\gamma}$ forms a monogon, it can be lifted to a series of forward flips $\mathbb{A} \xrightarrow{\gamma_1} \mathbb{A}^{\sharp}_{\gamma_1} \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_n} \mathbb{A}^{\sharp}_{\gamma_n}$ in $\mathrm{EG}^{\circ}(\mathbf{S}_{\mathrm{w}})$.

Contents

Bounded T-structures and Hearts

Definition

A t-structure on a triangulated category $\mathcal{D}$ is a torsion pair $(\mathcal{P}, \mathcal{P}^\perp)$ s.t. $\mathcal{P}[1] \subset \mathcal{P}$.
It is bounded if $M[\gg 0] \in \mathcal{P}$ and $M[\ll 0] \in \mathcal{P}^\perp$ for any object M.

Definition/Lemma (BBD)

A heart $\mathcal{H}$ of $\mathcal{D}$ is $\mathcal{P}^\perp[1] \cap \mathcal{P}$ for some bounded t-structure $(\mathcal{P}, \mathcal{P}^\perp)$, which is an abelian category.

- A heart is finite if it is generated by a finite set of simples $\operatorname{Sim}\mathcal{H}$.
- The (total) exchange graph $\operatorname{EG}(\mathcal{D})$ is an oriented graph whose vertices are all hearts in $\mathcal{D}$ and edges correspond to forward simple tiltings between them.

Triangulated Category associated with a wDMS

We call $\mathcal{D}(\mathbf{S}_{\mathrm{w}})$ a triangulated category associated with a weighted DMS $\mathbf{S}_{\mathrm{w}}$ if it satisfies the following three conditions:

- There is a bijection between $\widetilde{\mathrm{CA}}(\mathbf{S}_{\mathrm{w}})$ and some set of objects in $\mathcal{D}(\mathbf{S}_{\mathrm{w}})$ in the sense that for each graded closed arc $\widetilde{\eta} \in \widetilde{\mathrm{CA}}(\mathbf{S}_{\mathrm{w}})$, there is a unique object $X_{\widetilde{\eta}}$ associated to $\widetilde{\eta}$.
- For any three graded closed arcs $\widetilde{\eta}_1, \widetilde{\eta}_2, \widetilde{\eta}_3 \in \widetilde{\mathrm{CA}}(\mathbf{S}_{\mathrm{w}})$ forming a contractible triangle on $\mathbf{S}_{\mathrm{w}}$, there is a corresponding triangle

$$X_{\widetilde{\eta}_1} \to X_{\widetilde{\eta}_2}[a] \to X_{\widetilde{\eta}_3}[a+b] \to X_{\widetilde{\eta}_1}[a+b+c] = X_{\widetilde{\eta}_1}[1]$$

  in $\mathcal{D}(\mathbf{S}_{\mathrm{w}})$, where $a = \mathrm{index}_{z_1}(\widetilde{\eta}_1, \widetilde{\eta}_2)$, $b = \mathrm{index}_{z_2}(\widetilde{\eta}_2, \widetilde{\eta}_3)$ and $c = \mathrm{index}_{z_3}(\widetilde{\eta}_3, \widetilde{\eta}_1)$.
- For $\mathbb{A}_0^* = \{\widetilde{\eta}_1, \widetilde{\eta}_2, \ldots, \widetilde{\eta}_n\}$ which is the dual to a mixed-angulation $\mathbb{A}_0$, there is a finite heart $\mathcal{H}_{\mathbb{A}}$ associated to $\mathbb{A}_0$ which is finitely generated by simples $X_{\widetilde{\eta}_1}.X_{\widetilde{\eta}_2}, \ldots, X_{\widetilde{\eta}_n} \in \mathcal{H}_{\mathbb{A}} \subseteq \mathcal{D}(\mathbf{S}_{\mathrm{w}})$.

Isomorphism between Exchange Graphs

- $\mathcal{H}_0$: the canonical heart that is associated with the initial mixed-angulation $\mathbb{A}_0$.
- $\mathrm{EG}^\circ(\mathcal{D})$: a connected component of $\mathrm{EG}(\mathcal{D})$ which consists of all hearts that can be reached by repeated simple tiltings from $\mathcal{H}_0$.

Then there is a corollary of the three conditions of $\mathcal{D}(\mathbf{S}_\mathrm{w})$:

Lemma

There is an isomorphism between exchange graphs:

$$\mathrm{EG}^\circ(\mathcal{D}(\mathbf{S}_\mathrm{w})) \cong \mathrm{EG}^\circ(\mathbf{S}_\mathrm{w})\,. \tag{2}$$

The Verdier Quotient

Definition

The inclusion $\Sigma \subset \mathbf{S}_{\mathrm{w}}$ induces a short exact sequence of triangulated categories:

$$0 \longrightarrow \mathcal{D}(\Sigma) \longrightarrow \mathcal{D}(\mathbf{S}_{\mathrm{w}}) \longrightarrow \overline{\mathcal{D}}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}}) \longrightarrow 0,$$

where $\overline{\mathcal{D}}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}})$ is the Verdier quotient $\mathcal{D}(\mathbf{S}_{\mathrm{w}})/\mathcal{D}(\Sigma)$.

The main theorem

- $\mathrm{EG}(\overline{\mathcal{D}}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}}))$: the exchange graph of hearts in the quotient category.
- $\mathrm{EG}^{\bullet}(\overline{\mathcal{D}}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}}))$: the full subgraph of $\mathrm{EG}(\overline{\mathcal{D}}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}}))$ with vertices in the form of $\mathcal{H} \cap \mathcal{D}(\Sigma)$, where $\mathcal{H}$ is in the connected component $\mathrm{EG}^{\circ}(\mathcal{D}(\mathbf{S}_{\mathrm{w}}))$, called the principal part.

Theorem

There is an isomorphism between oriented graphs:

$$\mathrm{EG}^{\bullet}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}}) \cong \mathrm{EG}^{\bullet}(\overline{\mathcal{D}}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}})). \tag{3}$$

Contents

Further studies

- Take the topological Fukaya category to be a model of $\mathcal{D}(\mathbf{S}_{\mathrm{w}})$, and show that $\overline{\mathcal{D}}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}}) \cong \mathcal{D}(\overline{\mathbf{S}}_{\overline{\mathrm{w}}})$.
- Consider the stability conditions and quadratic differentials in the case of the collapsed surface $\overline{\mathbf{S}}_{\overline{\mathrm{w}}}$.

End

Thank you!