Entropy of cluster DT transformations and the finite-tame-wild trichotomy of acyclic quivers

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Finite-tame-wild trichotomy

Let Q be an acyclic quiver.

Then, the representation type of Q is either *finite*, *tame*, or *wild*.

Let $A_Q = (a_{ij})$ be the Cartan matrix of Q:

$$a_{ij} := \begin{cases} 2 & \text{if } i = j, \\ -\#\{i \to j \text{ in } Q\} & \text{else.} \end{cases}$$

Then, the representation type of Q is

- finite iff A_Q is positive definite,
- tame iff A_Q is positive semi-definite,
- wild iff A_Q is indefinite.

Nielsen–Thurston classification

For a real oriented surface Σ with $\chi(\Sigma) < 0$, mapping class $\phi \in MC(\Sigma) = \text{Homeo}^+/\text{homotopy}$ is either

- *periodic* (finite order in $MC(\Sigma)$),
- *reducible* (there exist a curve system on Σ fixed by ϕ),
- *pseudo-Anosov* (represented by a pseudo-Anosov homeo. f)

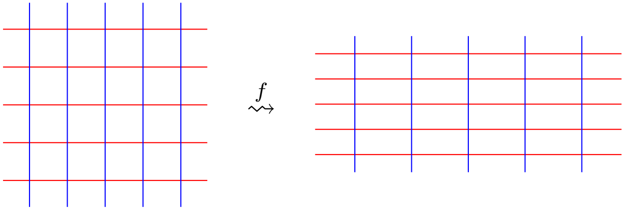
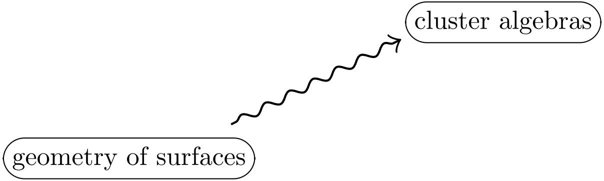


Illustration of this work



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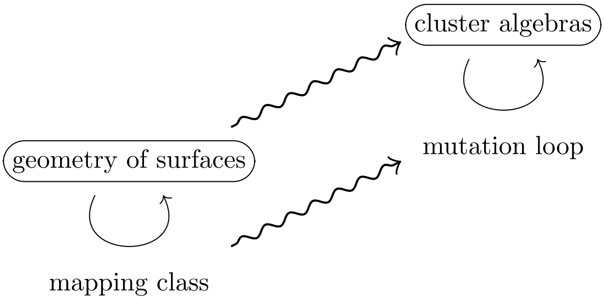

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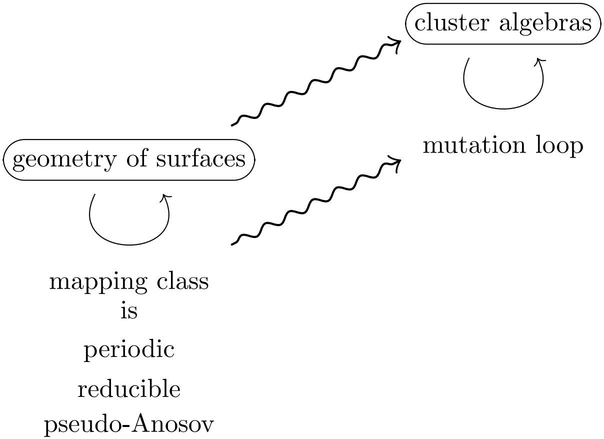

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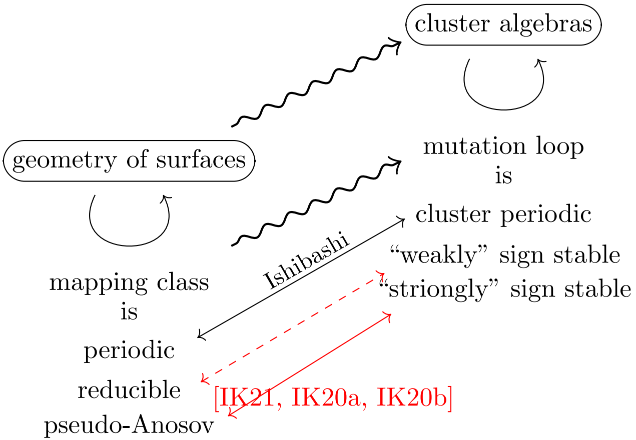

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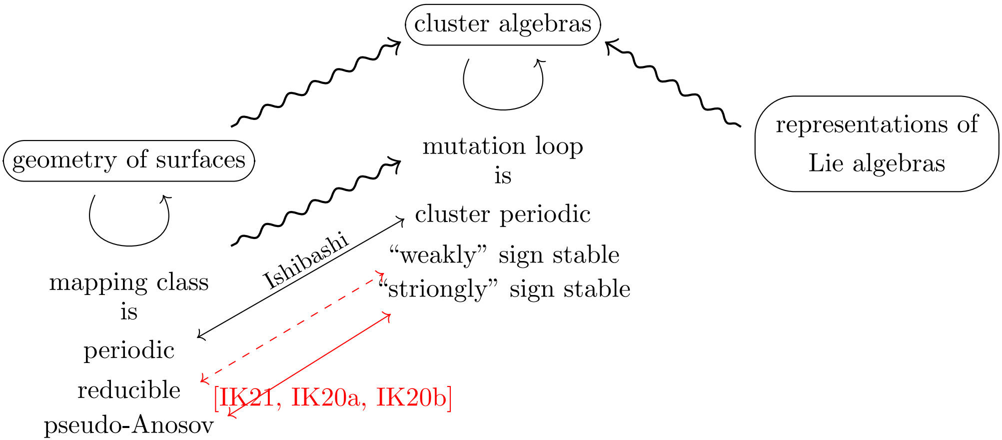

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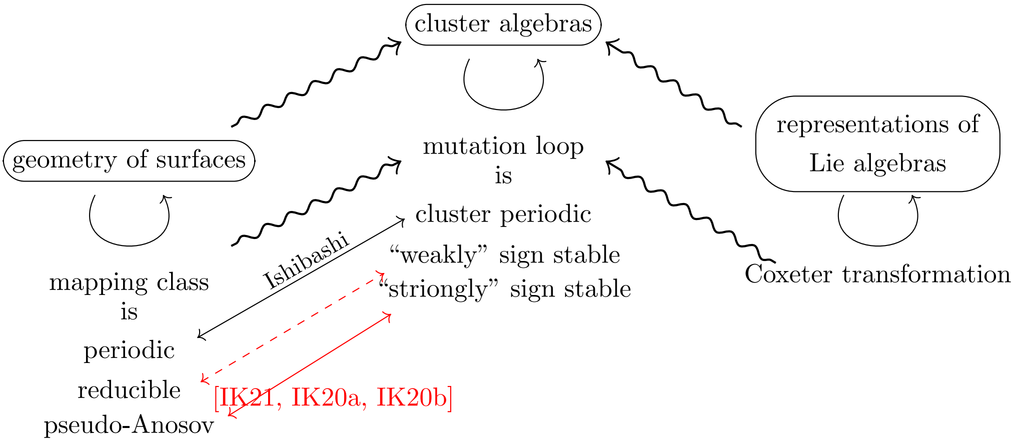

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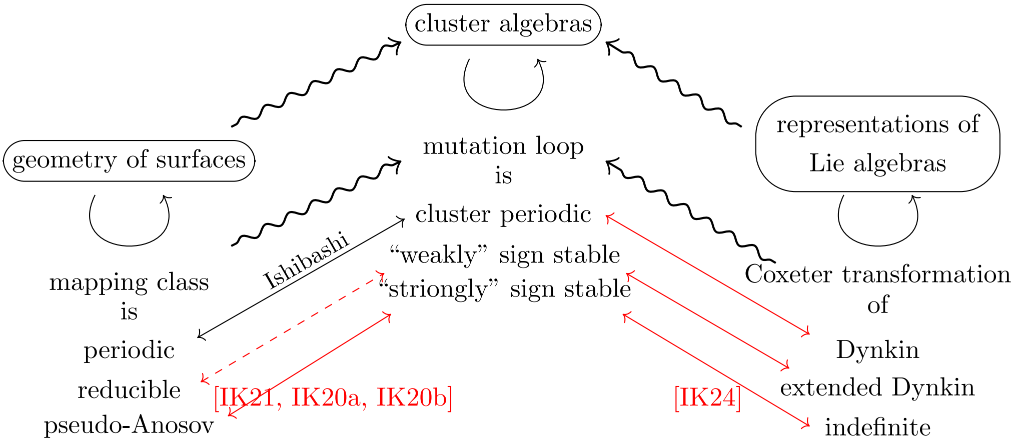

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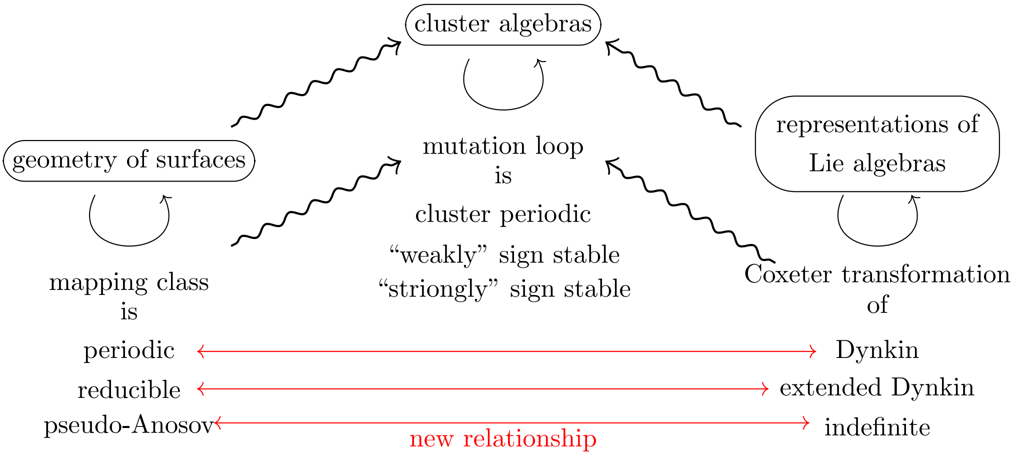


Mutation loop

Let Q be a quiver with vertex set I. A sequence

$$\gamma : Q = Q_0 \xrightarrow{k_0} Q_1 \xrightarrow{k_1} \cdots \xrightarrow{k_{h-1}} Q_h$$

of mutations is called a *(representation path of a) mutation loop* at Q if $Q_h = Q$ (up to relabeling by I).

Mutation loop

Let Q be a quiver with vertex set I. A sequence

$$\gamma : Q = Q_0 \overset{k_0}{-\!\!-\!\!-} Q_1 \overset{k_1}{-\!\!-\!\!-} \cdots \overset{k_{h-1}}{-\!\!-\!\!-} Q_h$$

of mutations is called a *(representation path of a)* *mutation loop* at Q if $Q_h = Q$ (up to relabeling by I).

$$Q = \begin{array}{c} 1 \\ \uparrow \\ 2 \\ \downarrow \\ 3 \end{array} \overset{\mu_2}{\longleftrightarrow} \begin{array}{c} 1 \\ \downarrow \\ 2 \\ \uparrow \\ 3 \end{array} \overset{\mu_1}{\longleftrightarrow} \begin{array}{c} 1 \\ \uparrow \\ 2 \\ \uparrow \\ 3 \end{array} \overset{\mu_3}{\longleftrightarrow} \begin{array}{c} 1 \\ \uparrow \\ 2 \\ \downarrow \\ 3 \end{array} = Q$$

Therefore, $\gamma : Q \overset{2}{-\!\!-\!\!-} Q_1 \overset{1}{-\!\!-\!\!-} Q_2 \overset{3}{-\!\!-\!\!-} Q$ is a mutation loop.

Sign of a mutation loop

For each mutation at $k \in I$, we assign the PL isomorphism

$$\mu_k : \mathbb{R}^I \to \mathbb{R}^I, \quad (x_i)_{i \in I} \mapsto (x'_i)_{i \in I}, \quad x'_i = \begin{cases} -x_k & \text{if } i = k, \\ x_i + [\operatorname{sgn}(x_k) b_{ik}]_+ x_k & \text{else.} \end{cases}$$

(cf. It is nothing but the *tropicalized cluster $\mathcal{X}$-transformation.*)

A *sign* $\boldsymbol{\epsilon}_\gamma = (\epsilon_0, \ldots, \epsilon_{h-1})$ of a mutation loop $\gamma : Q \overset{k_0}{\longrightarrow} \cdots \overset{k_{h-1}}{\longrightarrow} Q$ at $w \in \mathbb{R}^I$ is defined as

$$\epsilon_r := \text{the sign of the } k_r\text{-th coordinate of } \mu_{k_{r-1}} \cdots \mu_{k_0}(w)$$

Example of sign of a mutation loop

$$\gamma : \; Q = \begin{array}{c} 1 \\ \uparrow \\ 2 \\ \downarrow \\ 3 \end{array} \quad \overset{\mu_2}{\longleftrightarrow} \quad \begin{array}{c} 1 \\ \downarrow \\ 2 \\ \uparrow \\ 3 \end{array} \quad \overset{\mu_1}{\longleftrightarrow} \quad \begin{array}{c} 1 \\ \uparrow \\ 2 \\ \uparrow \\ 3 \end{array} \quad \overset{\mu_3}{\longleftrightarrow} \quad \begin{array}{c} 1 \\ \uparrow \\ 2 \\ \downarrow \\ 3 \end{array} = Q$$

$$w := \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \overset{-}{\longmapsto} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \overset{-}{\longmapsto} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \overset{+}{\longmapsto} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \mu_\gamma(w)$$

Therefore, $\boldsymbol{\epsilon}_\gamma(w) = (-, -, +)$

Sign stability

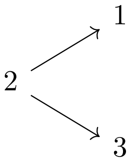
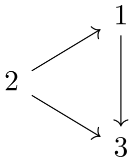
Definition

Let $\Omega \subset \mathbb{R}^I$ be an $\mathbb{R}_{>0}$-invariant subset. A mutation loop $\gamma : Q \xrightarrow{k_0} \cdots \xrightarrow{k_{h-1}} Q$ is *sign stable* on Ω if there is a sequence $\boldsymbol{\epsilon}^{\mathrm{stab}}_{\gamma,\Omega} \in \{+,-\}^h$ of strict signs such that for each $w \neq 0 \in \Omega$ there is $n_0 > 0$ satisfying

$$\boldsymbol{\epsilon}_\gamma(\mu^n_\gamma(w)) = \boldsymbol{\epsilon}^{\mathrm{stab}}_{\gamma,\Omega}$$

for all $n \geq n_0$. We call $\boldsymbol{\epsilon}^{\mathrm{stab}}_{\gamma,\Omega}$ the *stable sign* of γ on Ω.

(Counter) example of sign stable mutation loops

Q		
γ	$(2,1,3)$	$(2,1,3)$
$\boldsymbol{\epsilon}_\gamma(w)$	$(+,+,+)$	$(+,+,+)$
$\boldsymbol{\epsilon}_\gamma(\mu_\gamma(w))$	$(-,-,-)$	$(-,-,-)$
$\boldsymbol{\epsilon}_\gamma(\mu_\gamma^2(w))$	$(-,-,-)$	$(-,-,-)$
$\boldsymbol{\epsilon}_\gamma(\mu_\gamma^3(w))$	$(+,+,+)$	$(-,-,-)$
$\boldsymbol{\epsilon}_\gamma(\mu_\gamma^4(w))$	$(-,-,-)$	$(-,-,-)$
$\boldsymbol{\epsilon}_\gamma(\mu_\gamma^5(w))$	$(-,-,-)$	$(-,-,-)$
$\boldsymbol{\epsilon}_\gamma(\mu_\gamma^6(w))$	$(+,+,+)$	$(-,-,-)$
$\boldsymbol{\epsilon}_\gamma(\mu_\gamma^7(w))$	$(-,-,-)$	$(-,-,-)$

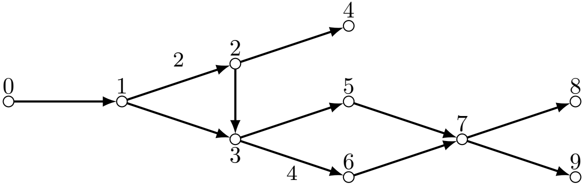
$$w := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3_{\geq 0}.$$

Cluster DT transformation of an acyclic quiver

Let Q be an acyclic quiver with admissible labeling $\pi : \{1, \ldots, N\} \to Q_0$.

Namely, $i < j$ if $\pi(i) \to \pi(j)$.

Example:



Then, the sequence

$$\gamma_\pi : Q \overset{1}{\rule[0.5ex]{2em}{0.4pt}} Q_1 \overset{2}{\rule[0.5ex]{2em}{0.4pt}} \cdots \overset{N}{\rule[0.5ex]{2em}{0.4pt}} Q_N$$

is mutation loop. ($i.e.$, $Q_N = Q$.)

Moreover, this mutation loop is a *cluster Donaldson–Thomas transformation.*

Namely, $\mu_{\gamma_\pi}(\mathbb{R}^N_{\geq 0}) = \mathbb{R}^N_{\leq 0}$.

Statement version 1

Theorem

Let Q and π be as the previous slide. Then, the representation type of Q is

1. *finite iff γ_π is not sign stable on $\mathbb{R}^N_{>0} \cup \mathbb{R}^N_{<0}$.*
2. *tame iff γ_π is sign stable on $\mathbb{R}^N_{>0} \cup \mathbb{R}^N_{<0}$ and the spectral radius of its Coxeter matrix is equal to 1.*
3. *wild iff γ_π is sign stable on $\mathbb{R}^N_{>0} \cup \mathbb{R}^N_{<0}$ and the spectral radius of its Coxeter matrix is larger than 1.*

Here, the Coxeter matrix Φ is defined as follows:

- $M := \#\{\text{direct paths in } Q \text{ from } i \text{ to } j\}$,
- $\Phi_\pi := -M_\pi^{-1} \cdot M_\pi^{\mathsf{T}}$.

Stable presentation matrix and cluster stretch factor

Let $\gamma : Q \xrightarrow{k_0} Q_1 \xrightarrow{k_1} \cdots \xrightarrow{k_{h-1}} Q_h = Q$.

Recall that $\mu_\gamma : \mathbb{R}^I \to \mathbb{R}^I$ is a piecewise-linear isomorphsm.

The "piecewise linearity" of μ_γ is controlled by its sign $\boldsymbol{\epsilon}_\gamma$.

Namely, for $\boldsymbol{\epsilon} \in \{+, -\}^h$, the set

$$\mathcal{C}_\gamma^{\boldsymbol{\epsilon}} := \{w \in \mathbb{R}^I \mid \boldsymbol{\epsilon}_\gamma(w) = \boldsymbol{\epsilon}\}$$

is a domain of linearity of μ_γ.

Let $E_\gamma^{\boldsymbol{\epsilon}}$ denote the presentation matrix of μ_γ on $\mathcal{C}_\gamma^{\boldsymbol{\epsilon}}$. For a sign-stable mutation loop γ with the stable sign $\boldsymbol{\epsilon}_\gamma^{\mathrm{stab}}$, we call the presentation matrix $E_\gamma^{\mathrm{stab}} := E_\gamma^{\boldsymbol{\epsilon}_\gamma^{\mathrm{stab}}}$ *stable presentation matrix*.

We call the spectral radius $\lambda_\gamma := \rho(E_\gamma^{\mathrm{stab}})$ of the stable presentation matrix *cluster stretch factor*.

Algebraic entropy of a sign-stable mutation loop

A mutation loop γ induces birational maps

$$\mu_\gamma^a : (\mathbb{C}^\times)^I \dashrightarrow (\mathbb{C}^\times)^I, \qquad \mu_\gamma^x : (\mathbb{C}^\times)^I \dashrightarrow (\mathbb{C}^\times)^I$$

by composing the corresponding cluster $\mathcal{A}$- and $\mathcal{X}$-transformations along γ.

Theorem (IK21)

If γ is sign-stable on $\mathbb{R}_{>0} \cup \mathbb{R}_{<0}$ and satisfies palindromicity property, then the algebraic entropies $h_{\mathrm{alg}}(\mu_\gamma^a)$ and $h_{\mathrm{alg}}(\mu_\gamma^x)$ are given by the logarithm of the cluster stretch factor:

$$h_{\mathrm{alg}}(\mu_\gamma^a) = h_{\mathrm{alg}}(\mu_\gamma^x) = \log(\lambda_\gamma)(= \rho(E_\gamma^{\mathrm{stab}})).$$

Categorical entropy of a sign-stable mutation loop

Theorem (K. 21)

If a mutation loop $\gamma : Q \xrightarrow{k_0} \cdots \xrightarrow{k_{h-1}} Q$ *is sign-stable on* $\mathbb{R}^I_{>0}$ *and satisfies palindromicity property induces a derived equivalence* F_γ *of a Ginzburg dg algebra* Γ *associated with the quiver* Q *with non-degenerate potential* W *so that the categorical entropies of the restrictions*

$$F_\gamma|_{\mathsf{per}} : \mathsf{per}(\Gamma) \to \mathsf{per}(\Gamma), \qquad F_\gamma|_{\mathsf{pvd}} : \mathsf{pvd}(\Gamma) \to \mathsf{pvd}(\Gamma)$$

are given by the logarithm of the cluster stretch factor:

$$h_T(F_\gamma|_{\mathsf{pvd}}) = h_0(F_\gamma|_{\mathsf{per}}) = \log(\lambda_\gamma), \quad (T \in \mathbb{R}).$$

Statement version 2

Theorem (IK24)

For an acyclic quiver Q with an admissible labeling π, if γ_π is sign stable then the stable presentation matrix coincides with the Coxeter matrix: $E^{\mathrm{stab}}_{\gamma_\pi} = \Phi_\pi$.

Therefore, we can update the statement version 1 as follows:

Theorem (IK24)

Let Q and π be as above. Then, the representation type of Q is

1. *finite iff γ_π is not sign-stable on* $\mathbb{R}^N_{>0} \cup \mathbb{R}^N_{<0}$.
2. *tame iff γ_π is sign-stable on* $\mathbb{R}^N_{>0} \cup \mathbb{R}^N_{<0}$ *and its entropies are equal to 0.*
3. *wild iff γ_π is sign-stable on* $\mathbb{R}^N_{>0} \cup \mathbb{R}^N_{<0}$ *and its entropies are larger than 0.*

Vanishing of entropy for finite and tame case

Theorem (IK24)

If Q is acyclic and its representation type is finite or tame, then the algebraic entropies of its any mutation loop is zero.

Future direction

We hope that the similar statement holds for any quivers with cluster DT transformations:
We expect that there exists the classes corresponds to

- cluster DT transformation is not sign-stable,
- cluster DT transformaiton is sign-stable and its entropies are equal to 0, and
- cluster DT transformaiton is sign-stable and its entropies are larger than 0,

respectively.