Hall algebra approach to ıquantum groups

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1. Background

Quantum groups(=QGs) and Hall algebras

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- Ringel'1990s, Green'1995, Xiao'1997, Bridgeland'2013, ... Hall algebra realization of QGs
- Kapranov'1997, Baumann-Kassel'2001, Schiffmann'2004, Dou-Jiang-Xiao'2012, Burban-Schiffmann'2013 Hall algebra realization of QGs in Drinfeld type presentation
- Lu-Peng'2016 Semi-derived Hall algebra, applied to two presentations

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Goal

- Construct a new kind of algebra, called Δ-Hall algebra
- Compare Δ-Hall algebra with derived Hall algebra and $\imath$Hall algebra
- Use Δ-Hall algebra to realize the $\imath$quantum group

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2. Δ-Hall algebra and derived Hall algebras

Δ-Hall number

Let $\mathcal{A}$ be a Hom-finite hereditary abelian category over $\mathbf{k} = \mathbb{F}_q$.

Definition

The Δ*-Hall number* for $A, B, M \in \mathcal{A}$ is defined as

$$\hat{F}_{AB}^{M} = \sum_{[L],[I],[N]} \mathbf{v}^{\langle L,I,N\rangle} \cdot \frac{a_L a_I a_N}{a_M} \cdot F_{LI}^{B} F_{NL}^{M} F_{IN}^{A}.$$

Here

$$\langle L, I, N\rangle = \langle L, I\rangle + \langle I, I\rangle + \langle I, N\rangle - \langle L, N\rangle,$$

and $\langle A, B\rangle = \dim \mathrm{Hom}_{\mathcal{A}}(A, B) - \dim \mathrm{Ext}^1_{\mathcal{A}}(A, B)$

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Three-cycle

Roughly speaking, the Δ-Hall number $\widehat{F}_{AB}^{M}$ calculates the number ($F_{LI}^{B}F_{NL}^{M}F_{IN}^{A}$) of three-cycles of exact sequences as follow:

$$\begin{array}{ccccc} & & I & & \\ & \nearrow & & \searrow & \\ & A & & B & \\ \nearrow & & & & \searrow \\ N & \leftarrow & M & \leftarrow & L \end{array}$$

Compare with Ringle's original Hall number F_{AB}^{M}:

$$0 \to B \to M \to A \to 0$$

Associativity

Proposition (Chen-Lin-R.)

For any objects $A, B, C, M \in \mathcal{A}$, the following equation holds:

$$\sum_{[X]} \widehat{F}_{AB}^{X} \widehat{F}_{XC}^{M} = \sum_{[Y]} \widehat{F}_{AY}^{M} \widehat{F}_{BC}^{Y}.$$

[Chen-Lin-Ruan] J. Chen, Y. Lin and S. Ruan. New realization of ıquantum groups via Δ-Hall algebras, J. Algebra 653 (2024), 378–403.

Green's formula

Proposition (Green)

The following formula holds in $\mathcal{A}$.

$$\sum_{[E]} F^E_{MN} F^E_{XY} \frac{1}{a_E} = \sum_{[A],[B],[C],[D]} q^{-\langle A,D\rangle} F^M_{AB} F^N_{CD} F^X_{AC} F^Y_{BD} \cdot \frac{a_A a_B a_C a_D}{a_M a_N a_X a_Y}.$$

[Green] J. Green. Hall algebras, hereditary algebras and quantum groups. Invent. Math. 120 (2) (1995), 361-377.

Δ-Hall algebra

Proposition (Chen-Lin-R.)

The $\mathbb{Q}(\mathbf{v})$*-vector space* $\mathcal{H}_\Delta(\mathcal{A})$ *with the basis* $\{[M] \mid [M] \in \mathrm{Iso}(\mathcal{A})\}$, *endowed with the multiplication defined by*

$$[A] * [B] = \sum_{[M]} \hat{F}_{AB}^{M} \cdot [M],$$

forms an associative algebra with the unit $[0]$, *called the* ***Δ-Hall algebra*** *of* $\mathcal{A}$.

1-periodic derived Hall algebras

Denote by

- $\mathcal{D}_1(\mathcal{A})$: the derived category of 1-periodic complexes on $\mathcal{A}$
- $\tilde{a}_M := |\mathrm{Aut}_{\mathcal{D}_1(\mathcal{A})}(M)|$
- $(A, B)_M := \{f \in \mathrm{Hom}_{\mathcal{D}_1(\mathcal{A})}(A, B) | \mathrm{Cone}(f) \cong M\}$
- $\{A, B\} := \dfrac{1}{|\mathrm{Hom}_{\mathcal{D}_1(\mathcal{A})}(A, B)|}$

Derived Hall algebra

Definition (Xu-Chen)

The *1-periodic derived Hall algebra* $\mathcal{DH}_1(\mathcal{A})$ is a $\mathbb{Q}(\mathbf{v})$-vector space with the basis $\{u_{[M]} \,|\, [M] \in \mathrm{Iso}(\mathcal{D}_1(\mathcal{A}))\}$, endowed with the multiplication defined by

$$u_{[A]} * u_{[B]} = \sum_{[M]} G_{AB}^{M} \cdot u_{[M]}$$

where

$$G_{AB}^{M} = \frac{\tilde{a}_M \cdot |(A,B)_M|}{\tilde{a}_A \cdot \tilde{a}_B} \sqrt{\frac{\{A,B\}\{M,M\}}{\{A,A\}\{B,B\}}}.$$

[Xu-Chen] F. Xu and X. Chen. Hall algebras of odd periodic triangulated categories, Algebr. Represent. Theory 16 (3) (2013), 673–687.

Δ-Hall algebra VS derived Hall algebra

Proposition (Chen-Lin-R.)

There is an algebra isomorphism

$$\Xi_{\mathcal{A}} : \mathcal{DH}_1(\mathcal{A}) \to \mathcal{H}_{\Delta}(\mathcal{A})$$

$$u_{[M]} \mapsto \frac{1}{a_M} \cdot [M].$$

3. Extended Δ-Hall algebra and $\imath$Hall algebra

ıHall algebra

- 1-periodic complexes: $M^\bullet = (M, d)$, $d^2 = 0 : M \to M$
- $\mathcal{C}_1(\mathcal{A})$: the category of 1-periodic complexes in $\mathcal{A}$
- $\mathcal{H}(\mathcal{C}_1(\mathcal{A}))$: Ringel-Hall algebra of $\mathcal{C}_1(\mathcal{A})$,

$$[M^\bullet] \diamond [N^\bullet] = \sum_{[L^\bullet] \in \mathrm{Iso}(\mathcal{C}_1(\mathcal{A}))} \frac{|\operatorname{Ext}^1(M^\bullet, N^\bullet)_{L^\bullet}|}{|\operatorname{Hom}(M^\bullet, N^\bullet)|} [L^\bullet]$$

- [Lu-Peng'2016], [Lu-Wang'2020] ıHall algebra ${}^\imath\widetilde{\mathcal{H}}(\mathcal{A})$: $\mathcal{H}(\mathcal{C}_1(\mathcal{A}))$ +quotient+localization+twist
- ${}^\imath\widetilde{\mathcal{H}}(\mathcal{A})$ has an (ıHall) basis given by

$$\{[M] * [K_\alpha] \mid [M] \in \mathrm{Iso}(\mathcal{A}), \alpha \in K_0(\mathcal{A})\}$$

Extended Δ-Hall algebras

Denote by

- $K_0(\mathcal{A})$: the Grothendieck group of $\mathcal{A}$, assume free
- $\widehat{M}$: the image of $M \in \mathcal{A}$ in $K_0(\mathcal{A})$
- $\widetilde{\mathcal{H}}_\Delta(\mathcal{A})$: the $\mathbb{Q}(\mathbf{v})$-vector space with the basis

$$\{[M][K_\alpha] \mid [M] \in \mathrm{Iso}(\mathcal{A}), \alpha \in K_0(\mathcal{A})\}$$

Extended Δ-Hall algebras

Proposition (Chen-Lin-R.)

The $\mathbb{Q}(\mathbf{v})$-vector space $\widetilde{\mathcal{H}}_{\Delta}(\mathcal{A})$ endowed with the multiplication defined by

$$[A][K_\alpha] * [B][K_\beta] = \sum_{[M]} \hat{F}_{AB}^{M} \cdot [M][K_{\frac{\widehat{A}+\widehat{B}-\widehat{M}}{2}+\alpha+\beta}],$$

in which $[A],[B] \in \mathrm{Iso}(\mathcal{A})$ and $\alpha, \beta \in K_0(\mathcal{A})$, forms an associative algebra with unit $[0][K_0]$, called the extended Δ-Hall algebra of $\mathcal{A}$.

The relation between extended Δ-Hall algebras and $\imath$Hall algebras

Proposition (Chen-Lin-R.)

There is an algebra isomorphism

$$\widetilde{\Xi}_{\mathcal{A}} : {}^{\imath}\widetilde{\mathcal{H}}(\mathcal{A}) \longrightarrow \widetilde{\mathcal{H}}_{\Delta}(\mathcal{A}).$$

$$[M] * [K_\alpha] \mapsto \mathbf{v}^{-\langle M,M\rangle}[M][K_\alpha]$$

4. New realizations of ıquantum groups

Notations for ıQG

- $C = (c_{ij})_{i,j\in I}$: symmetric generalized Cartan matrix
- $\mathfrak{g}$: Kac-Moody Lie algebra
- $[n] = [n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}$; $[m]_v^! = \prod_{j=1}^m [j]_v$
- $n \geq 1, i \in I$,

$$E_i^{(n)} = \frac{E_i^n}{[n]!}, \quad F_i^{(n)} = \frac{F_i^n}{[n]!}$$

The *universal quantum group* $\widetilde{\mathbf{U}} := \widetilde{\mathbf{U}}_{\mathbf{v}}(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be the $\mathbb{Q}(\mathbf{v})$-algebra generated by $E_i, F_i, \widetilde{K}_i, \widetilde{K}_i',\ i \in I$, where $\widetilde{K}_i, \widetilde{K}_i'$ are invertible, subject to the following relations for $i, j \in I$:

$$[E_i, F_j] = \delta_{ij} \frac{\widetilde{K}_i - \widetilde{K}_i'}{\mathbf{v} - \mathbf{v}^{-1}}, \qquad [\widetilde{K}_i, \widetilde{K}_j] = [\widetilde{K}_i, \widetilde{K}_j'] = [\widetilde{K}_i', \widetilde{K}_j'] = 0, \tag{1}$$

$$\widetilde{K}_i E_j = \mathbf{v}^{c_{ij}} E_j \widetilde{K}_i, \qquad \widetilde{K}_i F_j = \mathbf{v}^{-c_{ij}} F_j \widetilde{K}_i, \tag{2}$$

$$\widetilde{K}_i' E_j = \mathbf{v}^{-c_{ij}} E_j \widetilde{K}_i', \qquad \widetilde{K}_i' F_j = \mathbf{v}^{c_{ij}} F_j \widetilde{K}_i', \tag{3}$$

and the quantum Serre relations for $i \neq j \in I$,

$$\sum_{r=0}^{1-c_{ij}} (-1)^r E_i^{(r)} E_j E_i^{(1-c_{ij}-r)} = 0, \tag{4}$$

$$\sum_{r=0}^{1-c_{ij}} (-1)^r F_i^{(r)} F_j F_i^{(1-c_{ij}-r)} = 0. \tag{5}$$

Analogous as for $\widetilde{\mathrm{U}}$, the *quantum group* $\mathbf{U} := \mathbf{U}_{\mathbf{v}}(\mathfrak{g})$ is defined to be the $\mathbb{Q}(v)$-algebra generated by E_i, F_i, K_i, K_i^{-1}, $i \in I$, subject to the relations modified from (1)–(5) with $\widetilde{K}_i$ and $\widetilde{K}_i'$ replaced by K_i and K_i^{-1}, respectively.

The *universal ıquantum group* of split type $\widetilde{\mathbf{U}}^\imath$ is defined to be the $\mathbb{Q}(v)$-subalgebra of $\widetilde{\mathbf{U}}$ generated by

$$B_i = F_i + E_i\widetilde{K}_i', \qquad \Bbbk_i = \widetilde{K}_i\widetilde{K}_i', \quad \forall i \in I.$$

Let $\varsigma = (\varsigma_i)_{i\in I} \in (\mathbb{Q}(\mathbf{v})^\times)^I$. The *ıquantum group* of split type $\mathbf{U}^\imath := \mathbf{U}^\imath_\varsigma$ is the $\mathbb{Q}(v)$-subalgebra of $\mathbf{U}$ generated by

$$B_i = F_i + \varsigma_i E_i K_i^{-1}, \qquad \forall i \in I.$$

We consider the case

$$\varsigma_i = -v^{-2}, \qquad \forall i \in I, \tag{6}$$

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Main result

Theorem (Chen-Lin-R.)

Let Q be an arbitrary quiver without loops. We have the following commutative diagram of algebra homomorphisms

$$
\begin{array}{ccccc}
\langle \mathbb{k}_i + v^{-2} \rangle & \longrightarrow & \langle [K_\alpha] - 1 \rangle & \longrightarrow & \langle [K_\alpha] - 1 \rangle \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathbf{U}}^\imath_{|v=\mathbf{v}} & \xrightarrow{[\mathrm{LW2}]} & {}^\imath\widetilde{\mathcal{H}}(\mathbf{k}Q) & \xrightarrow{\cong} & \widetilde{\mathcal{H}}_\Delta(\mathbf{k}Q) \\
\Big\downarrow {\scriptstyle [\mathrm{LW1}]} & & {\scriptstyle [\mathrm{CLR}]} \Big\downarrow & & \Big\downarrow \\
\mathbf{U}^\imath_{|v=\mathbf{v}} & \xrightarrow{[\mathrm{CLR}]} & \mathcal{D}\mathcal{H}_1(\mathbf{k}Q) & \xrightarrow{\cong} & \mathcal{H}_\Delta(\mathbf{k}Q)
\end{array}
$$

Main result

Theorem (Chen-Lin-R.)

For any weighted projective line $\mathbb{X}_{\mathbf{k}}$, there exists a $\mathbb{Q}(\mathbf{v})$-algebra homomorphism

$$ {}^{Dr}\widetilde{\mathbf{U}}^{\imath}{}_{|v=\mathbf{v}} \xrightarrow[\text{[LR]}]{\text{[LRW]}} {}^{\imath}\widetilde{\mathcal{H}}(\mathbb{X}_{\mathbf{k}}) \xrightarrow{\cong} \widetilde{\mathcal{H}}_{\Delta}(\mathbb{X}_{\mathbf{k}}) $$

References I

[CLR] J. Chen, M. Lu and S. Ruan, *ıQuantum groups of split type via derived Hall algebras*, J. Algebra **610** (2022), 391-408.

[LR] M. Lu and S. Ruan. *ıHall algebras of weighted projective lines and quantum symmetric pairs.* Represent. Theory **28** (2024), 112–188.

[LRW] M. Lu, S. Ruan and W. Wang. *ıHall algebra of the projective line and q-Onsager algebra.* Trans. Amer. Math. Soc. **376** (2023), no. 2, 1475–1505.

[LW1] M. Lu and W. Wang. *Hall algebras and quantum symmetric pairs I: foundations*, Proc. London Math. Soc. **124** (3) (2022), no. 1, 1–82.

[LW2] M. Lu and W. Wang. *Hall algebras and quantum symmetric pairs of Kac-Moody type*, Adv. in Math. **430**, 109215, 2023.

Thank you!