A new approach to Auslander-Reiten formulae and almost split sequences in abelian categories

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1. For all $X \in \mathrm{Mod}\Lambda$, we have

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2. $\mathrm{Mod}\Lambda$ has an almost split sequence

$$0 \longrightarrow \mathrm{Hom}_\Gamma(\mathrm{Tr}M, I) \longrightarrow E \longrightarrow M \longrightarrow 0.$$

Remark

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2. Auslander's approach for the AR-formula involves Tor^1_Λ, not available in general abelian categories.

Objective

Provide a new approach to establish AR-formulae and AR-sequences in abelian categories with a Nakayama functor.

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In this case, $\nu\mathcal{P}$ is a subcategory of injective objects of $\mathcal{A}$.

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4. We need to pose some finiteness conditions on $\mathcal{P}$.

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1. *If* M *is indec non-injective, then* d^1 *is right minimal.*
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3. *For any $X \in \mathcal{A}$, we have*

$$D^2\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-}M, X) \cong D\,\overline{\mathrm{Hom}}_{\mathcal{A}}(X, M).$$

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2. Given $x \in Q_0$, put $P_x = \Lambda e_x$, where $e_x = \varepsilon_x + I$.

Lemma

$\mathcal{P}_\Lambda = \mathrm{add}\{P_x\langle i\rangle \mid x \in Q_0, i \in \mathbb{Z}\}$ *is Hom-finite Krull-Schmidt.*

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Theorem

We have a Nakayama functor

$$\nu = \mathfrak{D}\circ(-)^t : \mathcal{P}_\Lambda \longrightarrow \mathrm{GMod}\Lambda : P \mapsto \mathfrak{D}(P^t).$$