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Invariant subspaces of nilpotent operators. Level, mean and colevel: The triangle $\mathbb{T}(n)$

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A report on joint work with
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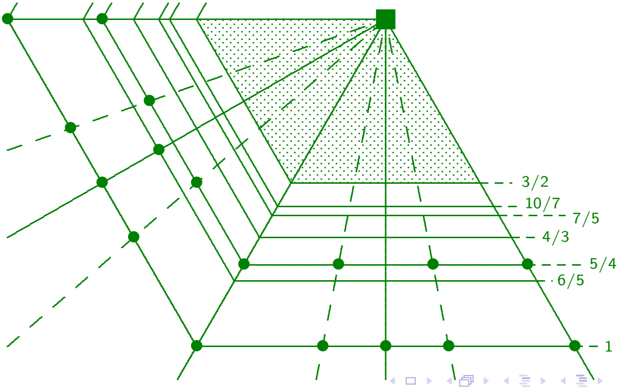
$\mathbb{T}(6)$:



Contents of my talk

Linear Algebra 3

LA1:
- vectorspaces V, finite dimensional over a field k
- subspaces $U \subset V$
- linear maps $T \colon U \to V$
- etc.

LA2:
- linear operators $T \colon V \to V$
- Jordan normal form
- etc.

LA3:
- linear operators with an invariant subspace (U, V)

$$\begin{cases} V & \text{is a finite dimensional vector space} \\ T \colon V \to V & \text{a linear operator, usually nilpotent, and} \\ U \subset V & \text{a subspace with } TU \subset U \end{cases}$$

Example (Pickets): Let $1 \leq m, 0 \leq \ell \leq m$ be integers.

- $V = k[T]/(T^m)$ has 1 Jordan block of size m
- $U = (T^{m-\ell})$ is the ℓ-dimensional kernel of T^ℓ
- We write $(U, V) = ([\ell], [m])$

Definition: $\mathcal{S}(n) = \{(U, V) : T^n = 0\}$ invariant subspaces of T^n-bounded linear operators

Related topics and applications

- Birkhoff 1938: Subgroups of abelian groups
  link to $\mathcal{S}(n)$: Gao, Külshammer, Kvamme, Psaroudakis '23
- Gorenstein-projective modules $\mathcal{S}(n) = \text{G-proj } U_2(k[T]/(T^n))$
  G-proj=smon (Xiuhua Luo, Pu Zhang '17)
- The Invariant Subspace Problem in functional analysis
  seems to be a theorem now (Per Enflo, Charles Neville '23)
- The algebraic Riccati equation in control theory
  ...determines the solution of ... two of the most fundamental problems in control theory [W]
- Persistance homology in topological data analysis
  differential complexes are graded square-zero operators...
- LA $\vee$ Calculus = pure math $\wedge$ applied math $\wedge$ applications
  ICOT conference series in Tunisia...

Basic invariants, the *pr*-triangle $\mathbb{T}(n)$

The non-zero object $(U, V) \in \mathcal{S}(n)$ has the invariants:

- $u = \dim U$
- $v = \dim V$, or equivalently, $w = \dim V/U$
- $b = \dim \operatorname{Ker} T = \#\{\text{Jordan blocks of } V\}$

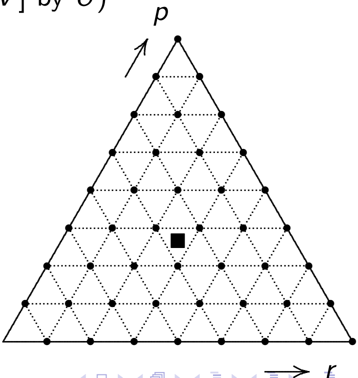
This yields the projective coordinates

- $q = v/b$ the mean (average size of Jordan blocks)
- $p = u/b$ the level (of a filling of $[V]$ by U)
- $r = w/b$ the colevel

The point $(p, r) \in \mathbb{R}^2$ occurs inside the triangle

- $p \geq 0$,
- $r \geq 0$,
- $q = p + r \leq n$.

Here is the *pr*-triangle $\mathbb{T}(8)$ with one picket at each lattice point $\neq (0, 0)$.



Theorem 1

Theorem 1. *Let $V = (V, T)$ be a linear operator and U a T-invariant subspace of V such that the pair (U, V) is indecomposable. Then either $U = 0$ (and V has only one Jordan block) or the dimension of U is at least the number of Jordan blocks of V.*

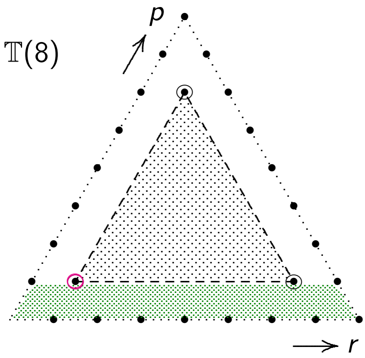
Theorem 1 (reformulation). *Suppose $X \in \mathcal{S}$ is indecomposable. Then either $uX = 0$ and $bX = 1$ (so X is a picket) or else $uX \geq bX$.*

Theorem 1 (reformulation). *Let $X = (U, V)$ be an object in $\mathcal{S}$. If $uX < bX$, then V has a direct decomposition $V = V_1 \oplus V_2$ with $TV_i \subset V_i$ for $i = 1, 2$, such that $U \subset V_1$ and $V_2 \neq 0$.*

Sparsity

Theorem 1. *Suppose $X \in \mathcal{S}$ is indecomposable. Then either $uX = 0$ and $bX = 1$ (so X is a picket) or else $uX \geq bX$.*

Theorem 2. *Suppose $(U, V) \in \mathcal{S}$ is indecomposable and such that the average size of the Jordan blocks of V is at most two. Then (U, V) is either a picket $([t], [m])$ with $m \leq 2$ or the indecomposable bipicket E with global space $[3, 1]$, subspace $[2]$ and factor $[2]$.*

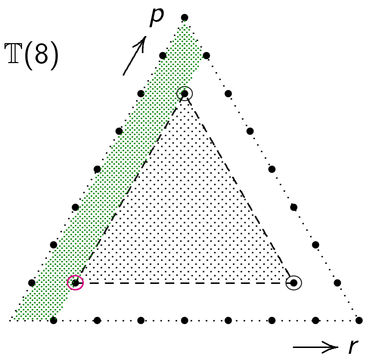


Duality

The duality D on $\mathcal{S}$ is given for $X = (U, V)$ by
$DX = ((V/U)^*, V^*)$.

$$uDX = wX, \qquad vDX = vX, \qquad wDX = uX, \qquad bDX = bX.$$

Theorem 1′. *Let X be an indecomposable object in $\mathcal{S}$. Then either $wX = 0$ (and X is a picket), or else $wX \geq bX$.*



The square of the Auslander-Reiten translation

Let $n \geq 1$. An object $X \in \mathcal{S}(n)$ is reduced if it has no non-zero projective direct summand. By $\tau = \tau_n$ denote the Auslander-Reiten translation in $\mathcal{S}(n)$. Write $\omega X = \dim \Omega V = n\,bX - v$.

Lemma: *If $X \in \mathcal{S}(n)$ is reduced, then so is $\tau^2 X$ and we have*

$$u\tau^2 X = wX,\ \omega\tau^2 X = uX,\ w\tau^2 X = \omega X,\ b\tau^2 X = bX.$$

Proof: An embedding $X = (U \subset V)$ gives rise to a short exact sequence in $\bmod k[T]/(T^n)$, hence to a triangle in the stable category:

$$U \to V \to W \to \Omega U \to$$

Recall that τX gives rise to the rotated triangle

$$V \to W \to \Omega U \to \Omega V \to$$

hence $\tau^2 X$ to the triangle

$$W \to \Omega U \to \Omega V \to \Omega W \to .$$

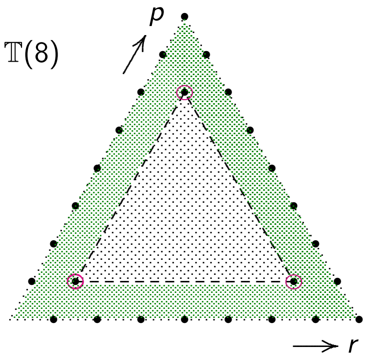
Moreover,

$$n\,bX = \dim PV = uX + wX + \omega X = wX + \omega X + uX = n\,b\tau^2 X.$$

The rotation

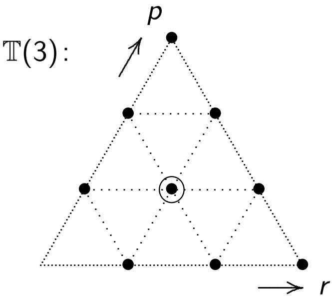
On $\mathbb{T}(n)$, the operator τ^2 gives rise to a rotation by 120° as it permutes the coordinates: $p \rightsquigarrow r \rightsquigarrow (n-q) \rightsquigarrow p$

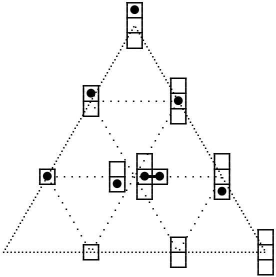
Theorem 1″. *Let X be an indecomposable object in $\mathcal{S}(n)$. Then either $vX = n\,bX$ (and X is a picket), or else $vX \leq (n-1)bX$.*



The case $n = 3$

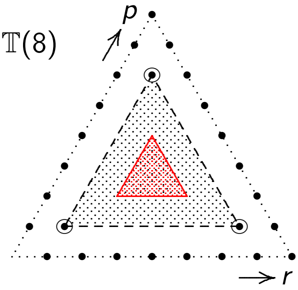
In the case where $n = 3$, Theorems 1, 1′, 1″ and 2 recover the full classification of the indecomposable objects in $\mathcal{S}(n)$: There are 9 pickets and the bipicket E.





Density: BTh-vectors

Theorem. *Any rational pr-vector with boundary distance at least 2 is a BTh-vector.*



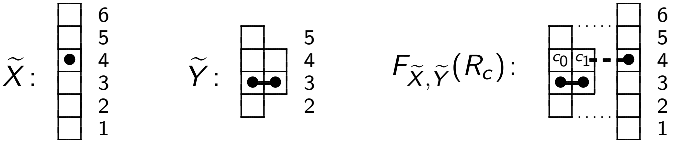
A $\mathbb{P}^1$-family $\{M_c : c \in \mathbb{P}^1\}$ is a set of pairwise non-isomorphic indecomposable objects with fixed values for uM_c, wM_c and bM_c, indexed by the elements $c = (c_0 : c_1)$ of the projective line $\mathbb{P}^1 = \mathbb{P}^1(k) = \{\text{one-dimensional subspaces in } k^2\}$.

A point $(p, r) \in \mathbb{T}(n)$ is a BTh-vector if there is an integer a such that for any natural number t there is a $\mathbb{P}^1$-family $\{M_c\}$ in $\mathcal{S}(n)$ with $uM_c = atp$, $wM_c = atr$ and $bM_c = at$.

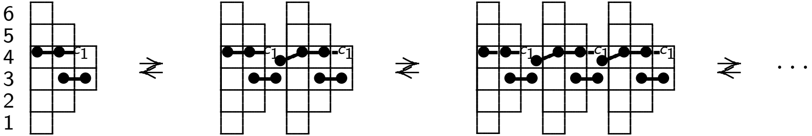
The center of $\mathbb{T}(6)$

In $\mathbb{T}(6)$, the point $(2, 2)$ is the only BTh-vector.

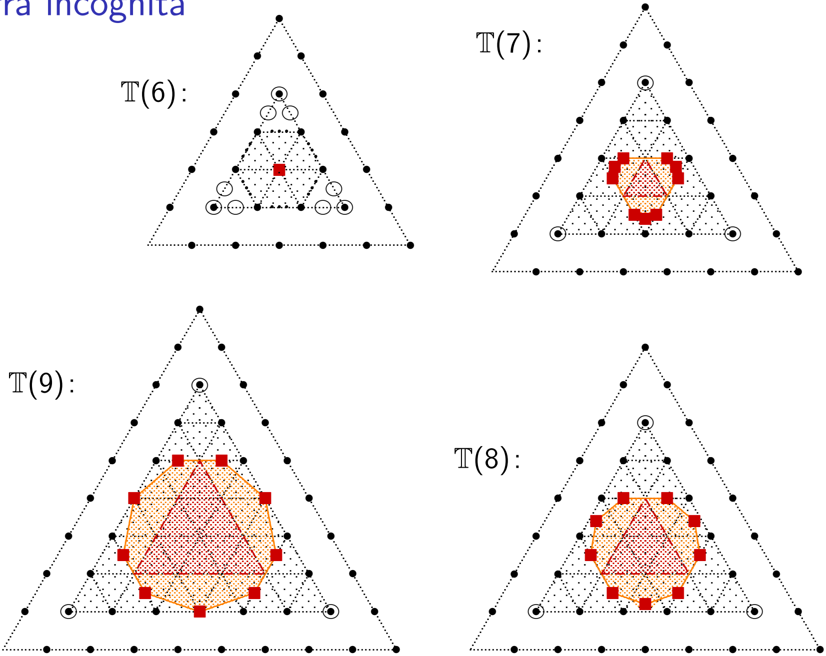
Consider the orthogonal pair $(\widetilde{X}, \widetilde{Y})$ in $\widetilde{\mathcal{S}}(6)$. It satisfies $\dim \operatorname{Ext}^1(\widetilde{X}, \widetilde{Y}) = 2$, hence gives rise to a $\mathbb{P}^1$-family.



Here are the extensions in case $c_0 \neq 0$ (where we assume $c_0 = 1$):



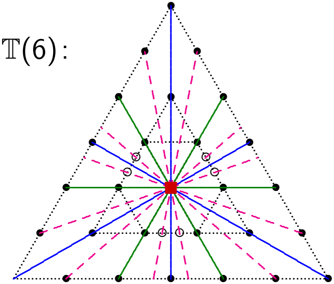
Terra incognita



The case $n = 6$: 12 lines

Theorem. *The pr-vector of each indecomposable object in $\mathcal{S}(6)$ lies on one of 12 central lines. In the variables p, r and $\omega = n - q$, the equations have the form $p = r$, $\omega = 2$ and $r = 2(\omega - 1)$.*

$\mathbb{T}(6)$:



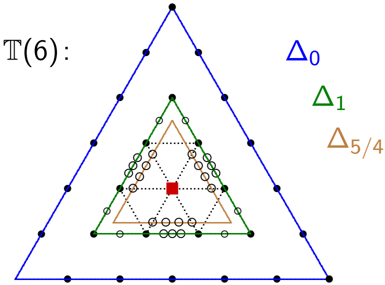
Necessity: Each non-central object X in an AR-component gives rise to an infinite sequence of objects in the same component on the half-line connecting $\mathrm{pr}(X)$ and the center.

The triangle support for $\mathcal{S}(6)$

Theorem. *There exists an increasing sequence* Ψ *of numbers in* $[0, 2)$ *which converges to* 2 *with the following properties.*

(a) Each indecomposable object in $\mathcal{S}(6)$ *occurs on the standard triangle* Δ_d *for some* $d \in \Psi$.

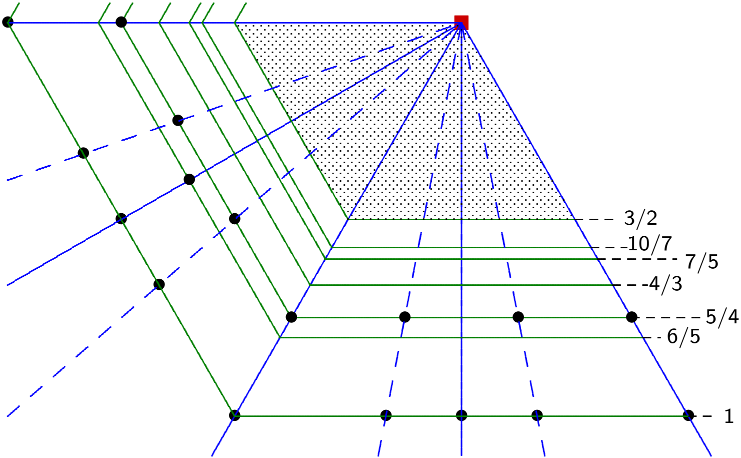
(b) Each triangle Δ_d *is support of finitely many indecomposable objects in* $\mathcal{S}(6)$.



Remark. In the Theorem, "standard triangles" can be replaced by "costandard triangles" or "hexagons".

The spider web for $n = 6$

Zooming into $\mathbb{T}(6)$ reveals the following picture:



Thank you!

Some literature

- N. Gao, J. Külshammer, S. Kvamme, Ch. Psaroudakis, *A functorial approach to monomorphism categories II: Indecomposables,* arXiv: 2303.07753, 2023, 46 pp.
- S. Kvamme, *An introduction to monomorphism categories,* to appear in Proc. Conf. Icra 20, arXiv: 2407.17147, 36 pp.
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- X. Luo, S. Zhu, *Auslander-Reiten translations in the monomorphism categories of exact categories,* arXiv: 2408.01359, 2024, 1–26
- C. M. Ringel, M. Schmidmeier, *The Auslander-Reiten translation in submodule categories,* Trans. Amer. Math. Soc. 2008, 691–716
- C. M. Ringel, M. Schmidmeier, *Invariant subspaces of nilpotent linear operators. Level, mean, colevel: The triangle* $\mathbb{T}(n)$, arXiv: 2405.18592