Stable categories of Gorenstein-projective modules over monomial algebras

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1. Motivation

- Λ : a finite dimensional algebra over a field K
- $\operatorname{mod}\Lambda$: the category of (finitely generated right) Λ-modules
- $\operatorname{proj}\Lambda$: the category of projctive Λ-modules

Definition (Enochs-Jenda 1995)

M_Λ : Gorenstein-projective (GP) $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ an acyclic complex of projective Λ-modules

$$P^\bullet : \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \to \cdots$$

s.t. (i) $\operatorname{Hom}_\Lambda(P^\bullet, \Lambda)$ is exact; and (ii) $M \cong \operatorname{Ker} d^0$ as Λ-modules

- $P^\bullet$: a complete resolution of M
- $\operatorname{Gproj}\Lambda$: the category of GP Λ-modules $\Longrightarrow$ $\operatorname{proj}\Lambda \subseteq \operatorname{Gproj}\Lambda \subseteq \operatorname{mod}\Lambda$
- $\operatorname{Gproj}\Lambda = \operatorname{mod}\Lambda \iff \Lambda$: self-injective

Definition

Λ : CM-free $\overset{\text{def}}{\Longleftrightarrow}$ $\operatorname{proj}\Lambda = \operatorname{Gproj}\Lambda$

- $\operatorname{gl.dim}\Lambda < \infty \Longrightarrow \Lambda$: CM-free

- From now on, assume Λ is monomial (i.e. $\Lambda = KQ/I$, where I is generated by paths)

$\Longrightarrow$ Λ is CM-finite (i.e. $\#\operatorname{ind}\operatorname{Gproj}\Lambda < \infty$), because any indecomposable non-projective GP Λ-module is of the form $p\Lambda$ for some non-zero non-trivial path p

Definition (Chen-Shen-Zhou 2018)

- A pair (p, q) of non-zero paths in Λ is perfect if the following are satisfied:
  1. p and q are both non-trivial with $t(p) = s(q)$ and satisfy $pq = 0$ in Λ
  2. If $pq' = 0$ for a non-zero path q' with $t(p) = s(q')$, then $q' = qq''$ for some path q''
  3. If $p'q = 0$ for a non-zero path p' with $t(p') = s(q)$, then $p' = p''p$ for some path p''
- $(p_1, \ldots, p_n, p_{n+1} = p_1)$: a perfect path sequence if (p_i, p_{i+1}) is perfect for $1 \leq i \leq n$
- A path in a perfect path sequence is called a perfect path

- $\mathbb{P}_\Lambda$: the set of perfect paths

Theorem (CSZ 2018)

$$\begin{array}{ccc} \mathbb{P}_\Lambda & \xleftrightarrow{1:1} & \{\text{indecomposable non-projective GP } \Lambda\text{-modules}\}/\cong \\ p & \longmapsto & p\Lambda \end{array}$$

- $\mathbb{P}_\Lambda$ is empty $\iff$ Λ is CM-free

Remark Perfect path sequences give rise to complete resolutions

• The stable category $\underline{\operatorname{Gproj}}\,\Lambda = \operatorname{Gproj}\Lambda/\operatorname{proj}\Lambda$ of $\operatorname{Gproj}\Lambda$ carries a structure of a triangulated category

Theorem (CSZ 2018)

TFAE

1. $\underline{\operatorname{Gproj}}\,\Lambda$ is a semisimple triangulated category
2. $\exists$ no overlap in Λ
3. $\underline{\operatorname{Gproj}}\,\Lambda \cong \prod_{1\leq i\leq n}(\operatorname{mod} k^{n_i}, \sigma^*)$, where $\sigma : k^{n_i} \to k^{n_i} \in \operatorname{Aut}\Lambda$ given by $\sigma(\lambda_1, \lambda_2, \ldots, \lambda_{n_i}) = (\lambda_2, \ldots, \lambda_{n_i}, \lambda_1)$

Remark $\exists$ no overlap in $\Lambda \iff$ there exists no non-trivial morphism in $\underline{\operatorname{Gproj}}\,\Lambda$

• Ringel (2013) and Lu-Zhu (2021) determined $\underline{\operatorname{Gproj}}\,\Lambda$ for Nakayama algebras and 1-Iwanaga-Gorenstein monomial algebras, respectively

• In any cases, $\underline{\operatorname{Gproj}}\,\Lambda \cong \underline{\operatorname{mod}}\,\Gamma$ for some self-injective Nakayama algebra Γ

Our aim is

to describe $\underline{\operatorname{Gproj}}\,\Lambda$ for more general cases

2. Stable categories of graded Gorenstien-projective modules

Definition

For $p \in \mathbb{P}_\Lambda$, the associated underlying cycle c_p is the shortest cycle c s.t. $p_1 \cdots p_n = c^l$ for some $l > 0$, where $(p = p_1, \ldots, p_n, p_{n+1} = p_1)$ is a perfect path sequence

- $\mathcal{C}(\Lambda)$: the set of equivalence classes (w.r.t. cyclic permutation) of underlying cycles

Definition

For p and $q \in \mathbb{P}_\Lambda$, we write $p \preceq q$ if $q = pr$ for some path r

- $(\mathbb{P}_\Lambda, \preceq)$ is a poset
- The Hasse quiver $H(\mathbb{P}_\Lambda, \preceq)$ is a disjoint union of linear quivers

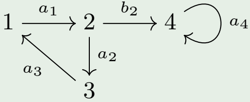
Definition

$p \in \mathbb{P}_\Lambda$: co-elementary $\overset{\text{def}}{\Longleftrightarrow}$ p is a sink in $H(\mathbb{P}_\Lambda, \preceq)$

- $\mathbb{E}_\Lambda^{\text{co}}$: the set of co-elementary paths

Example

Let $\Lambda = KQ/I$ be the monomial algebra given by



$$a_{1231} = a_{23123} = a_4^4 = 0$$

- The following are the minimal perfect path sequences

$$(a_1,\ a_{231},\ a_{23},\ a_{123},\ a_1),\ \ (a_4,\ a_4^3,\ a_4),\ \ (a_4^2,\ a_4^2)$$

- $\mathbb{P}_\Lambda = \{a_1,\, a_{231},\, a_{23},\, a_{123},\, a_4,\, a_4^2,\, a_4^3\}$
- $\mathcal{C}(\Lambda) = \{a_{123},\, a_4\}$, where $a_{123} = a_{231}$
- $H(\mathbb{P}_\Lambda, \preceq):\quad a_{123} \longrightarrow a_1 \qquad a_{231} \longrightarrow a_{23} \qquad a_4^3 \longrightarrow a_4^2 \longrightarrow a_4$
- $\mathbb{E}_\Lambda^{\mathrm{co}} = \{a_1, a_{23}, a_4\}$

Proposition-Definition

For $c \in \mathcal{C}(\Lambda)$, $\exists!\, r_1, \dots, r_n \in \mathbb{E}_\Lambda^{\mathrm{co}}$ s.t. $c = r_1 \cdots r_n$. We denote $|c| := n$.

- Consider $\Lambda = KQ/I$ as a positively graded algebra by defining $\deg a = 1$ for $a \in Q_1$

- $\mathrm{mod}^{\mathbb{Z}}\Lambda$: the category of graded Λ-modules

Recall For $M, N \in \mathrm{mod}^{\mathbb{Z}}\Lambda$, $\mathrm{Hom}_{\Lambda}^{\mathbb{Z}}(M, N) := \{f \in \mathrm{Hom}_{\Lambda}(M, N) \mid f(M_i) \subseteq N_i \text{ for } i\}$

- $\mathrm{proj}^{\mathbb{Z}}\Lambda$: the category of graded projective Λ-modules

- $\mathrm{Gproj}^{\mathbb{Z}}\Lambda$: the category of graded GP Λ-modules

- $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda = \mathrm{Gproj}^{\mathbb{Z}}\Lambda/\mathrm{proj}^{\mathbb{Z}}\Lambda$: the stable category of $\mathrm{Gproj}^{\mathbb{Z}}\Lambda$

- Lu-Zhu (2021) observed that $\mathrm{ind}\,\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda = \{p\Lambda(i) \mid p \in \mathbb{P}_{\Lambda}, i \in \mathbb{Z}\}$

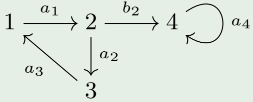
- For $c = r_1 \cdots r_n \in \mathcal{C}(\Lambda)$ with $r_i \in \mathbb{E}_\Lambda^{\mathrm{co}}$, we define

$$\mathbb{P}_\Lambda(c) := \{p \in \mathbb{P}_\Lambda \mid r_1 \preceq p\} \quad \text{and} \quad T_c := \bigoplus_{p \in \mathbb{P}_\Lambda(c)} p\Lambda$$

- Define $T := \bigoplus_{c \in \mathcal{C}(\Lambda)} \bigoplus_{0 \leq i < l(c)} T_c(i) \in \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$

Example

Let $\Lambda = KQ/I$ be as in the first example:



$$a_{1231} = a_{23123} = a_4^4 = 0$$

- $H(\mathbb{P}_\Lambda, \preceq)$: $\quad a_{123} \longrightarrow a_1 \qquad a_{231} \longrightarrow a_{23} \qquad a_4^3 \longrightarrow a_4^2 \longrightarrow a_4$

- $\mathcal{C}(\Lambda) = \{a_{123},\, a_4\}$

$\Longrightarrow \; T = a_1\Lambda \oplus a_{123}\Lambda \oplus a_1\Lambda(1) \oplus a_{123}\Lambda(1) \oplus a_1\Lambda(2) \oplus a_{123}\Lambda(2) \oplus a_4\Lambda \oplus a_4^2\Lambda \oplus a_4^3\Lambda$

- $\mathcal{C}(\Lambda) = \{a_{231},\, a_4\}$

$\Longrightarrow \; T = a_{23}\Lambda \oplus a_{231}\Lambda \oplus a_{23}\Lambda(1) \oplus a_{231}\Lambda(1) \oplus a_{23}\Lambda(2) \oplus a_{231}\Lambda(2) \oplus a_4\Lambda \oplus a_4^2\Lambda \oplus a_4^3\Lambda$

Theorem (Honma-U 2024)

1. T is a tilting object of $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$, namely,

$$\text{(i) } \underline{\mathrm{Hom}}_{\Lambda}^{\mathbb{Z}}(T, \Sigma^i T) = 0 \text{ for } i \neq 0; \qquad \text{(ii) } \mathrm{thick}\, T = \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$$

2. $\underline{\mathrm{End}}_{\Lambda}^{\mathbb{Z}} T \;\cong\; \prod_{c \in \mathcal{C}(\Lambda)} (K\mathbb{A}_c)^{(l(c))}$, where $\mathbb{A}_c : 1 \to 2 \to \cdots \to |\mathbb{P}_{\Lambda}(c)|$

3. $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \cong \prod_{c \in \mathcal{C}(\Lambda)} \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)^{(l(c))}$ as triangulated categories

Example

Let $\Lambda = KQ/I$ be as in the first example. Fix $\mathcal{C}(\Lambda) = \{a_{123},\, a_4\}$

- $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \;\cong\; \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_{a_{123}})^{(3)} \times \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_{a_4})$

[$H(\mathbb{P}_{\Lambda}, \preceq)$: $\quad a_{123} \longrightarrow a_1 \qquad a_{231} \longrightarrow a_{23} \qquad a_4^3 \longrightarrow a_4^2 \longrightarrow a_4$]

Remark The theorem describes the graded singularity category $\mathcal{D}_{\mathrm{sg}}(\mathrm{mod}^{\mathbb{Z}}\Lambda)$ of Λ when Λ is Iwanaga-Gorenstein and in particular improves a result of Lu-Zhu (2021) for Iwanaga-Gorenstein monomial algebras

Proposition (LZ 2021)

The forgetful functor $F : \operatorname{mod}^{\mathbb{Z}}\Lambda \to \operatorname{mod}\Lambda$ induces a G-covering

$$\tilde{F}_G : \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda \to \underline{\operatorname{Gproj}}\,\Lambda$$

in the sense of Asashiba (2011), where G is the cyclic group generated by the automorphism $(1) : \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda \to \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda$

- Thanks to Asashiba (2011), we obtain an equivalence $H : \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda/(1) \xrightarrow{\sim} \underline{\operatorname{Gproj}}\,\Lambda$ that makes the following diagram commute

$$\begin{array}{ccc} \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda & \xrightarrow{\quad \tilde{F}_G \quad} & \underline{\operatorname{Gproj}}\,\Lambda \\ & {\scriptstyle P}\searrow \qquad \nearrow{\scriptstyle H}\,\sim & \\ & \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda/(1) & \end{array}$$

- It follows from Section 2 that

1. $\underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda = \prod_{c\in\mathcal{C}(\Lambda)} \prod_{0\le i<l(c)} \operatorname{thick} T_c(i)$
2. $\operatorname{thick} T_c(i) \cong \mathcal{D}^{\mathrm{b}}(\operatorname{mod} K\mathbb{A}_c)$ for $c \in \mathcal{C}(\Lambda)$ and $i \in \mathbb{Z}$

- $\operatorname{thick} T_c(i) = (\operatorname{thick} T_c)(i)$ for $c \in \mathcal{C}(\Lambda)$ and $i \in \mathbb{Z}$
- $P(\operatorname{thick} T_c(i)) = P(\operatorname{thick} T_c)$ for $c \in \mathcal{C}(\Lambda)$ and $i \in \mathbb{Z}$

- $\underline{\mathrm{Gproj}}\,\Lambda \cong \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1) = \prod_{c\in\mathcal{C}(\Lambda)} P(\operatorname{thick} T_c)$, where $P(\operatorname{thick} T_c) = \operatorname{thick} P(T_c)$

Lemma

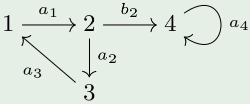
1. $\operatorname{thick} T_c(i) = \operatorname{thick} T_c(j)$ in $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \iff i \equiv j \pmod{l(c)}$ for $c \in \mathcal{C}(\Lambda)$ and $i, j \in \mathbb{Z}$
2. For $c \in \mathcal{C}(\Lambda)$, the restriction of $P : \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \to \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1)$ to $\operatorname{thick} T_c$ induces a G_c-covering
$$P_c : \operatorname{thick} T_c \to P(\operatorname{thick} T_c)$$
where G_c is the cyclic group generated by the induced automorphism $(l(c)) : \operatorname{thick} T_c \to \operatorname{thick} T_c$
3. For $c \in \mathcal{C}(\Lambda)$, $P(\operatorname{thick} T_c) \cong \operatorname{thick} T_c/(l(c)) \cong \mathcal{D}^{\mathrm{b}}(\operatorname{mod} K\mathbb{A}_c)/\tau^{|c|}$, where τ is the Auslander-Reiten translation for $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} K\mathbb{A}_c)$

Threoem (Honma-U 2024)

$$\begin{aligned}\underline{\mathrm{Gproj}}\,\Lambda &\cong \prod_{c\in\mathcal{C}(\Lambda)} \mathcal{D}^{\mathrm{b}}(\operatorname{mod} K\mathbb{A}_c)/\tau^{|c|} \\ &\cong \prod_{c\in\mathcal{C}(\Lambda)} \underline{\operatorname{mod}}\, K\left(1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow |c| \;(\text{with } |c| \to 1)\right)\Big/ R^{|\mathbb{P}_\Lambda(c)|+1}\end{aligned}$$

Example

Let $\Lambda = KQ/I$ be as in the first example:



$$a_{1231} = a_{23123} = a_4^4 = 0$$

- One computes

(1) $\mathcal{C}(\Lambda) = \{a_{123},\, a_4\}$ with $|a_{123}| = 2,\ \ |a_4| = 1$

(2) $|\mathbb{P}_\Lambda(a_{123})| = 2,\ |\mathbb{P}_\Lambda(a_4)| = 3$

- $\underline{\mathrm{Gproj}}\,\Lambda \cong \underline{\mathrm{mod}}\,K\big(1 \rightleftarrows 2\big)/R^3 \times \underline{\mathrm{mod}}\,K[x]/(x^4)$

Remark The theorem describes the singularity categories $\mathcal{D}_{\mathrm{sg}}(\mathrm{mod}\,\Lambda)$ of Iwanaga-Gorenstein monomial algebras Λ. Moreover, it recovers results of Ringel (2013), Chen-Shen-Zhou (2018), and Lu-Zhu (2021).

Thank you!