Triangulated structures induced by mutations

a simultaneous generalization of twin cotorsion pairs and mutation pairs

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Motivation

In this talk,

“subcategory” = additive full subcategory which is closed under isomorphisms and direct summands.

Fact

1. (Iyama-Yoshino'08) $\mathcal{D}$-mutation pair $(\mathcal{Z}, \mathcal{Z})$ where $\mathcal{Z}$ is extension closed induces a triangulated category $\mathcal{Z}/[\mathcal{D}]$.
2. (Simões-Pauksztello'20) $\langle \mathcal{M} \rangle$-mutation pair $(\mathcal{Z}, \mathcal{Z})$ which satisfies some conditions induces a tri. cat. $\mathcal{Z}$.
3. (Nakaoka-Palu'19) A Frobenius extriangulated (ET) cat. $\mathcal{C}$ induces a tri. cat. $\mathcal{C}/[\mathrm{Proj}\,\mathcal{C}]$.
4. (Nakaoka'18) A Hovey twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ induces a triangulated cat. $\mathcal{T} \cap \mathcal{U}/[\mathcal{S} \cap \mathcal{V}]$.

Motivation

These triangulated structures are induced by mutations.

1. (Iyama-Yoshino'08) Mutations of cluster-tilting / silting subcategory.
2. (Simões-Pauksztello'20) Mutations of simple-minded system / collection.
3. (Nakaoka-Palu'19) Special case of ❶ in ET categories.
4. (Nakaoka'18) The shift functor is exactly the right mutation in ❷.

Question

Is there a simultaneous generalization of previous four facts?

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4. (Nakaoka'18) The shift functor is exactly the right mutation in 2.

Question

Is there a simultaneous generalization of previous four facts?

Answer

It is exactly a mutation triple satisfying (MT4).

Mutation triple

Definition

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an ET cat. and $\mathcal{S}, \mathcal{Z}, \mathcal{V}$ be subcategories of $\mathcal{C}$.

- A triplet $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ is called mutation triple (MT) if it satisfies the conditions from (MT1) to (MT3).
- We introduce an additional condition for MTs: (MT4).

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- We introduce an additional condition for MTs: (MT4).

Example

1. (IY'08) $(\mathcal{D}, \mathcal{Z}, \mathcal{D})$ is a MT satisfying (MT4).
2. (SP'20) $(\langle \mathcal{M}[1] \rangle, \mathcal{Z}, \langle \mathcal{M}[-1] \rangle)$ is a MT satisfying (MT4).
3. (NP'19) $(\mathrm{Proj}\,\mathcal{C}, \mathcal{C}, \mathrm{Proj}\,\mathcal{C})$ is a MT satisfying (MT4).
4. (Nak'18) $(\mathcal{S}, \mathcal{T} \cap \mathcal{U}, \mathcal{V})$ is a MT satisfying (MT4).

Main theorem

Theorem 1 [I, arXiv'24:2406.19625v2]

Mutation triple $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ induces a pretriangulated category $\mathcal{Z}/[\mathcal{S} \cap \mathcal{V}]$.

Theorem 2 [I, arXiv'24]

Moreover, if $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ satisfies (MT4),
then it induces a tri. cat. $\mathcal{Z}/[\mathcal{S} \cap \mathcal{V}]$.

Corollary

1. (IY'08) $\mathcal{Z}/[\mathcal{D}]$ is a tri. cat.
2. (SP'20) $\mathcal{Z}$ is a tri. cat.
3. (NP'19) $\mathcal{C}/[\mathrm{Proj}\,\mathcal{C}]$ is a tri. cat.
4. (Nak'18) $\mathcal{T} \cap \mathcal{U}/[\mathcal{S} \cap \mathcal{V}]$ is a tri. cat.

Relative structure

Let $\mathcal{I} \subset \mathcal{C}$ be a subcategory. For $f \colon X \to Y$,

- f : $\mathcal{I}$-monic $\iff$ $\mathcal{C}(f, \mathcal{I}) \colon \mathcal{C}(Y, \mathcal{I}) \to \mathcal{C}(X, \mathcal{I})$ is surjective.
- f : $\mathcal{I}$-epic $\iff$ $\mathcal{C}(\mathcal{I}, f) \colon \mathcal{C}(\mathcal{I}, X) \to \mathcal{C}(\mathcal{I}, Y)$ is surjective.

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Definition [Herschend-Liu-Nakaoka'21, Dräxler-Reiten-Smalø-Solberg-Keller'99]

Let $\mathcal{I} \subset \mathcal{C}$ be a subcategory.

- $\mathbb{E}^{\mathcal{I}}(Y,X) = \{\delta \in \mathbb{E}(Y,X); X \xrightarrow{f} E \xrightarrow{g} Y \overset{\delta}{\dashrightarrow} X$ where f is $\mathcal{I}$-monic.$\}$
- $\mathbb{E}_{\mathcal{I}}(Y,X) = \{\delta \in \mathbb{E}(Y,X); X \xrightarrow{f} E \xrightarrow{g} Y \overset{\delta}{\dashrightarrow} X$ where g is $\mathcal{I}$-epic.$\}$

We define $\mathfrak{s}^{\mathcal{I}}$ (resp. $\mathfrak{s}_{\mathcal{I}}$) by $\mathfrak{s}|_{\mathbb{E}^{\mathcal{I}}}$ (resp. $\mathfrak{s}|_{\mathbb{E}_{\mathcal{I}}}$).

Proposition [Herschend-Liu-Nakaoka'21]

$(\mathcal{C},\mathbb{E}^{\mathcal{I}},\mathfrak{s}^{\mathcal{I}})$ and $(\mathcal{C},\mathbb{E}_{\mathcal{I}},\mathfrak{s}_{\mathcal{I}})$ are ET cat.

Definition [Zhou-Zhu'18]

Let $\mathcal{I} \subset \mathcal{Z}$ be subcategories of $\mathcal{C}$.

1. $\mathcal{I} \subset \mathcal{Z}$ is strongly covariantly finite if
   ${}^{\forall}Z \in \mathcal{Z}$ ${}^{\exists}\mathfrak{s}$-inflation $Z \xrightarrow{f} I^Z$ where f is a left $\mathcal{I}$-approximation.
2. Dually, we define strongly contravariantly finite.
3. $\mathcal{I} \subset \mathcal{Z}$ is strongly functorially finite if
   $\mathcal{I} \subset \mathcal{Z}$ is both strongly cov.fin. and strongly cont.fin.

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Definition (MT1)

Let $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ be a triplet of subcat. of $\mathcal{C}$.

(MT1) (i) $\mathcal{S} \cap \mathcal{Z} = \mathcal{Z} \cap \mathcal{V}$, denoted by $\mathcal{I}$.
(ii) $\mathcal{I} \subset \mathcal{Z}$ is strongly funct.fin.

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(ii) $\mathcal{I} \subset \mathcal{Z}$ is strongly funct.fin.

Example : (MT1)

1. (IY'08) From $\mathcal{D} \subset \mathcal{Z}$, $\mathcal{I} = \mathcal{D}$.
2. (SP'20) From $\mathcal{Z} \subset {}^{\perp}\mathcal{M}[-1], \mathcal{M}[1]^{\perp}$, $\mathcal{I} = 0$.
3. (NP'19) $\mathcal{I} = \mathrm{Proj}\,\mathcal{C}$, which is strongly funct. fin. in $\mathcal{C}$.
4. (Nak'18) Because $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is concentric, $\mathcal{S} \cap \mathcal{Z} = \mathcal{Z} \cap \mathcal{V} = \mathcal{I}$.

Let $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ be a triplet of subcat. satisfying (MT1).

- For a subcat. $\mathcal{X}$, $\mathcal{X}/[\mathcal{I}]$ is denoted by $\underline{\mathcal{X}}$.
- From (MT1), for $Z \in \mathcal{Z}$, there exists the following $\mathfrak{s}^{\mathcal{I}}$-tri. and fix it.

$$Z \to I^Z \to Z\langle 1\rangle \overset{\lambda^Z}{\dashrightarrow} Z \quad (I^Z \in \mathcal{I})$$

Lemma 1

1. For $z\colon Z_1 \to Z_2$ in $\mathcal{Z}$,

$$^{\exists}z'\colon Z_1\langle 1\rangle \to Z_2\langle 1\rangle \text{ s.t. } \begin{array}{ccc} Z_1\langle 1\rangle & \overset{\lambda^{Z_1}}{\dashrightarrow} & Z_1 \\ {\scriptstyle z'}\downarrow & \circlearrowright & \downarrow{\scriptstyle z} \\ Z_2\langle 1\rangle & \overset{\lambda^{Z_2}}{\dashrightarrow} & Z_2 \end{array} \text{ (we denote } z' \text{ by } z\langle 1\rangle.)$$

2. $\langle 1\rangle$ induces a functor $\langle 1\rangle : \underline{\mathcal{Z}} \to \underline{\mathcal{C}}$ and does not depend on the choices of $\mathfrak{s}^{\mathcal{I}}$-tri. up to iso.

Dually, we define a functor $\langle -1\rangle : \underline{\mathcal{Z}} \to \underline{\mathcal{C}}$ by the following $\mathfrak{s}_{\mathcal{I}}$-tri.

$$Z \overset{\lambda_Z}{\dashrightarrow} Z\langle -1\rangle \to I_Z \to Z \quad (I_Z \in \mathcal{I})$$

Example : $\langle 1\rangle, \langle -1\rangle$

1. (IY'08) $\langle 1\rangle$ (resp. $\langle -1\rangle$) is called left (resp. right) mutation there.
2. (SP'20) Since $\mathcal{I} = 0$, $\langle 1\rangle = [1]$.
3. (NP'19) $\langle 1\rangle$ (resp. $\langle -1\rangle$) is so-called cosyzygy (resp. syzygy) functor.
4. (Nak'18) $\langle 1\rangle$ is defined in the same way.

Notations

Let $\mathcal{X}, \mathcal{Y}$ be subcat.

- $\mathrm{Cone}_{\mathbb{E}}(\mathcal{X}, \mathcal{Y})$
  $:= \left\{ Z \in \mathcal{C} \;\middle|\; \begin{array}{l} \text{There exists an } \mathfrak{s}\text{-tri.} \\ \qquad X \to Y \to Z \dashrightarrow X \text{ where } X \in \mathcal{X}, Y \in \mathcal{Y} \end{array} \right\}.$
- $\mathrm{CoCone}_{\mathbb{E}}(\mathcal{X}, \mathcal{Y})$
  $:= \left\{ Z' \in \mathcal{C} \;\middle|\; \begin{array}{l} \text{There exists an } \mathfrak{s}\text{-tri.} \\ \qquad X \dashrightarrow Z' \to X \to Y \text{ where } X \in \mathcal{X}, Y \in \mathcal{Y} \end{array} \right\}.$
- $\mathcal{Z}\langle 1 \rangle := \mathrm{Cone}_{\mathbb{E}^{\mathcal{I}}}(\mathcal{Z}, \mathcal{I})$
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Definition

Let $(\mathcal{S},\mathcal{Z},\mathcal{V})$ be a triplet of subcat. satisfying (MT1).

(MT2) (i) $\mathbb{E}^{\mathcal{I}}(\mathcal{S},\mathcal{Z})=0$ and $\mathbb{E}_{\mathcal{I}}(\mathcal{S},\mathcal{Z}\langle -1\rangle)=0$.

(ii) $\mathbb{E}_{\mathcal{I}}(\mathcal{Z},\mathcal{V})=0$ and $\mathbb{E}^{\mathcal{I}}(\mathcal{Z}\langle 1\rangle,\mathcal{V})=0$.

We use the following notations.

- $\widetilde{\mathcal{U}} := \mathrm{CoCone}_{\mathbb{E}^{\mathcal{I}}}(\mathcal{Z}, \mathcal{S})$
- $\widetilde{\mathcal{T}} := \mathrm{Cone}_{\mathbb{E}_{\mathcal{I}}}(\mathcal{V}, \mathcal{Z})$

For $U \in \widetilde{\mathcal{U}}$, there exists the following $\mathfrak{s}^{\mathcal{I}}$-tri. and fix it.

$$U \xrightarrow{h^U} \sigma U \to S^U \dashrightarrow U \quad (\sigma U \in \mathcal{Z}, S^U \in \mathcal{S})$$

(For $Z \in \mathcal{Z} \subset \widetilde{\mathcal{U}}$, we choose $Z = \sigma Z$ and $\mathrm{id}_Z = h^Z$.)

Lemma 2

Let $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ be a triplet of subcat. satisfying (MT1) and (MT2).
Then $- \circ \underline{h^U}\colon \underline{\mathcal{Z}}(\sigma U, \mathcal{Z}) \to \underline{\widetilde{\mathcal{U}}}(U, \mathcal{Z})$ is a natural iso.

Corollary

σ induces a functor $\sigma\colon \underline{\widetilde{\mathcal{U}}} \to \underline{\mathcal{Z}}$, which is a left adjoint functor of the inclusion functor $i\colon \underline{\mathcal{Z}} \to \underline{\widetilde{\mathcal{U}}}$.

On the other hand, for $T \in \widetilde{\mathcal{T}}$, there exists the following $\mathfrak{s}_{\mathcal{I}}$-tri. and fix it.

$$T \dashrightarrow V_T \to \omega T \xrightarrow{h_T} T \quad (\omega T \in \mathcal{Z}, V_T \in \mathcal{V})$$

Dually, we may define a functor $\omega\colon \underline{\widetilde{\mathcal{T}}} \to \underline{\mathcal{Z}}$, which is a right adjoint functor of the inclusion functor $j\colon \underline{\mathcal{Z}} \to \underline{\widetilde{\mathcal{T}}}$.

Example : σ

1. (IY'08) Since $\mathcal{Z} = \mathcal{Z}\langle 1\rangle$, $\sigma|_{\mathcal{Z}\langle 1\rangle} = \mathrm{Id}$.
2. (SP'20) For $Z \in \mathcal{Z}$, $\sigma(Z[1])$ is defined by choosing the following tri.

$$M \xrightarrow{m} Z[1] \to \sigma(Z[1]) \to M[1] \quad (m : \text{minimal right } \langle \mathcal{M}\rangle\text{-approx.})$$

   Note that $(\langle \mathcal{M}[1]\rangle, \mathcal{Z}, \langle \mathcal{M}[-1]\rangle)$ is a MT and $M[1] \in \langle \mathcal{M}[1]\rangle$.

3. (NP'19) Since $\mathcal{Z} = \widetilde{\mathcal{U}} = \mathcal{C}$, $\sigma = \mathrm{Id}$.
4. (Nak'18) σ is defined in the same way.

Let $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ be a triplet of subcat.

Definition

Assume that $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ satisfies (MT1) and (MT2).

(MT3) (i) $\mathrm{Cone}_{\mathbb{E}^{\mathcal{I}}}(\mathcal{Z}, \mathcal{Z}) \subset \widetilde{\mathcal{U}}$ and $\mathcal{S}, \mathcal{Z}$ are closed under extensions in $\mathbb{E}^{\mathcal{I}}$.
(ii) $\mathrm{CoCone}_{\mathbb{E}_{\mathcal{I}}}(\mathcal{Z}, \mathcal{Z}) \subset \widetilde{\mathcal{T}}$ and $\mathcal{Z}, \mathcal{V}$ are closed under ext. in $\mathbb{E}_{\mathcal{I}}$.

Definition

$(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ is a mutation triple if it satisfies (MT1), (MT2) and (MT3).

Definition

1. $\Sigma := \sigma \circ \langle 1 \rangle \colon \underline{\mathcal{Z}} \to \underline{\mathcal{Z}}$ is called right mutation.
2. $\Omega := \omega \circ \langle -1 \rangle \colon \underline{\mathcal{Z}} \to \underline{\mathcal{Z}}$ is called left mutation.

Definition

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Example : Σ, Ω

1. (IY'08)
   $\Sigma = \langle 1 \rangle$ (resp. $\Omega = \langle -1 \rangle$) is called left (resp. right) mutation there.
2. (SP'20) Σ (resp. Ω) is called right (resp. left) mutation there.
3. (NP'19)
   $\Sigma = \langle 1 \rangle$ (resp. $\Omega = \langle -1 \rangle$) is a so-called cosyzygy (resp. syzygy) functor.
4. (Nakaoka'18) Σ (resp. Ω) is defined in the same way.

Let $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ be a MT.

- For $a \colon X \to Y$ in $\mathcal{Z}$, there exists the following comm. diag. in $\mathbb{E}^{\mathcal{I}}$ from [Liu-Nakaoka'19].

$$\begin{array}{ccccccc} X & \xrightarrow{\begin{bmatrix} a \\ i^X \end{bmatrix}} & Y \oplus I^X & \xrightarrow{b} & C^a & \overset{\widetilde{\delta}}{\dashrightarrow} & X \\ \| & \circlearrowright & \downarrow{\scriptstyle [0\ 1]} & \circlearrowright & \downarrow{\scriptstyle c^a} & \circlearrowright & \| \\ X & \xrightarrow{i^X} & I^X & \xrightarrow{p^X} & X\langle 1 \rangle & \overset{\lambda^X}{\dashrightarrow} & X \end{array}$$

- $\color{red}{\nabla} = \left(\begin{array}{l} \text{seq. in } \underline{\mathcal{Z}} \text{ iso. to one in} \\ \{X \xrightarrow{\underline{a}} Y \xrightarrow{\underline{h^{C^a} \circ b}} \sigma C^a \xrightarrow{\sigma(\underline{c^a})} \Sigma X \mid a : \text{morph. in } \mathcal{Z}\}. \end{array} \right)$

We define $\color{red}{\Delta}$, dually.

Proposition

Let $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ be a MT.
Then $(\underline{\mathcal{Z}}, \Sigma, \nabla)$ is a right triangulated category in Beligiannis-Reiten'07.

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[Sketch of proof] (RT0) and (RT1) are clear.

(RT2) Let $b' = h^{C^a} \circ b$. Then a standard right tri. of b'

$$Y \xrightarrow{\underline{b'}} \sigma C^a \longrightarrow \sigma C^{b'} \longrightarrow \Sigma Y \text{ is iso. to the right rotated seq.}$$

$$Y \xrightarrow{\underline{b'}} \sigma C^a \xrightarrow{\sigma(\underline{c^a})} \Sigma X \xrightarrow{\Sigma(\underline{a})} \Sigma Y.$$

(RT3) Let $X_i, Y_i \in \mathcal{Z}$, $\delta_i \in \mathbb{E}^{\mathcal{I}}$ for $i = 1, 2$.

$$\begin{array}{ccccccc} X_1 & \xrightarrow{a_1} & Y_1 & \xrightarrow{b_1} & U_1 & \overset{\delta_1}{\dashrightarrow} & X_1 \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle x} \\ X_2 & \xrightarrow{a_2} & Y_2 & \xrightarrow{b_2} & U_2 & \overset{\delta_1}{\dashrightarrow} & X_2 \end{array} \Rightarrow \begin{array}{ccccccc} X_1 & \xrightarrow{\underline{a_1}} & Y_1 & \xrightarrow{\underline{h^{U_1} b_1}} & \sigma U_1 & \xrightarrow{\sigma(\underline{c^{a_1}})} & \Sigma X_1 \\ \downarrow{\scriptstyle \underline{x}} & & \downarrow{\scriptstyle \underline{y}} & & \downarrow{\scriptstyle \sigma(\underline{u})} & & \downarrow{\scriptstyle \Sigma\underline{x}} \\ X_2 & \xrightarrow{\underline{a_2}} & Y_2 & \xrightarrow{\underline{h^{U_2} b_2}} & \sigma U_2 & \xrightarrow{\sigma(\underline{c^{a_2}})} & \Sigma X_2 \end{array}$$

(RT4) $U_1 \xrightarrow{a} U_2 \xrightarrow{b} U_3 \overset{\delta}{\dashrightarrow} U_1 : \mathfrak{s}^{\mathcal{I}}$-tri. ($U_i \in \mathrm{Cone}_{\mathbb{E}^{\mathcal{I}}}(\mathcal{Z},\mathcal{Z})$ for $1 \le i \le 3$)

Then $\sigma U_1 \xrightarrow{\sigma(\underline{a})} \sigma U_2 \xrightarrow{\sigma(\underline{b})} \sigma U_3 \to \Sigma(\sigma U_1)$ is a right tri.

Dually, one can show the following proposition.

Proposition

Let $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ be a MT.
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By checking the compatibility of right triangles and left triangles, we get Theorem 1.

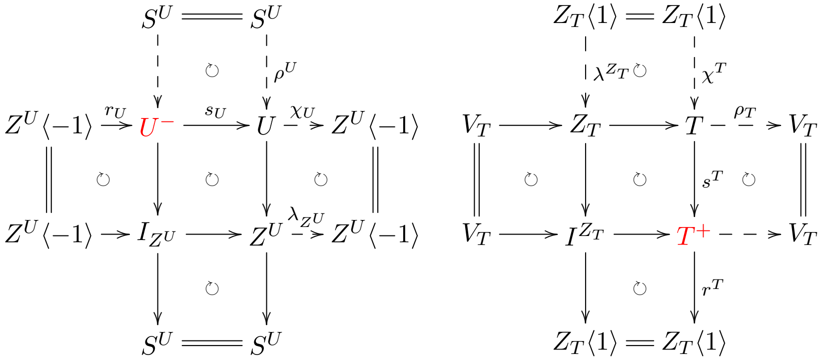
Theorem 1

Let $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ be a MT.
Then $(\underline{\mathcal{Z}}, \Sigma, \Omega, \nabla, \Delta)$ is a pretriangulated category.

Before we define (MT4), we introduce new cones and cocones.

- $\widetilde{\mathcal{U}}^- = \mathrm{CoCone}_{\mathbb{E}^{\mathcal{I}}}(\mathcal{I}, \mathcal{S}) \subset \widetilde{\mathcal{U}}$.
- $\widetilde{\mathcal{T}}^+ = \mathrm{Cone}_{\mathbb{E}_{\mathcal{I}}}(\mathcal{V}, \mathcal{I}) \subset \widetilde{\mathcal{T}}$.

For $U \in \widetilde{\mathcal{U}}$ and $T \in \widetilde{\mathcal{T}}$, $U^- \in \widetilde{\mathcal{U}}^-$ and $T^+ \in \widetilde{\mathcal{T}}^+$ are defined by:



Lemma

1. $(\cdot)^- : \underline{\widetilde{\mathcal{U}}} \to \underline{\widetilde{\mathcal{U}}^-}$ is a right adjoint functor of $i^- : \underline{\widetilde{\mathcal{U}}^-} \to \underline{\widetilde{\mathcal{U}}}$.
2. $(\cdot)^+ : \underline{\widetilde{\mathcal{T}}} \to \underline{\widetilde{\mathcal{T}}^+}$ is a left adjoint functor of $i^+ : \underline{\widetilde{\mathcal{T}}^+} \to \underline{\widetilde{\mathcal{T}}}$.

Lemma

1. $(\cdot)^{-}: \underline{\widetilde{\mathcal{U}}} \to \underline{\widetilde{\mathcal{U}}^{-}}$ is a right adjoint functor of $i^{-}: \underline{\widetilde{\mathcal{U}}^{-}} \to \underline{\widetilde{\mathcal{U}}}$.
2. $(\cdot)^{+}: \underline{\widetilde{\mathcal{T}}} \to \underline{\widetilde{\mathcal{T}}^{+}}$ is a left adjoint functor of $i^{+}: \underline{\widetilde{\mathcal{T}}^{+}} \to \underline{\widetilde{\mathcal{T}}}$.

Definition

Let $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ be a MT.

(MT4) (i) $\underline{\mathcal{Z}\langle 1\rangle^{-}} \subset \underline{\widetilde{\mathcal{T}}^{+}}$.

(ii) $\underline{\mathcal{Z}\langle -1\rangle^{+}} \subset \underline{\widetilde{\mathcal{U}}^{-}}$.

- Note that if $\underline{\widetilde{\mathcal{U}}^{-}} = \underline{\widetilde{\mathcal{T}}^{+}}$, then (MT4) holds. (it is denoted by (MT4^{+})).
- (MT4) is equivalent to $\underline{\mathcal{Z}\langle 1\rangle^{-}} = \underline{\mathcal{Z}\langle -1\rangle^{+}}$.

Example : (MT4)

1. (IY'08) $\underline{\mathcal{Z}\langle 1\rangle^-} = \underline{\mathcal{Z}\langle -1\rangle^+} = 0$.
2. (SP'20)
   - $\widetilde{\mathcal{U}}^- = \mathrm{CoCone}_{\mathbb{E}}(0, \langle \mathcal{M}[1]\rangle) = \langle \mathcal{M}\rangle$.
   - $\widetilde{\mathcal{T}}^+ = \mathrm{Cone}_{\mathbb{E}}(\langle \mathcal{M}[-1]\rangle, 0) = \langle \mathcal{M}\rangle$.
3. (NP'19)
   - $\widetilde{\mathcal{U}}^- = \mathrm{CoCone}_{\mathbb{E}}(\mathrm{Proj}\,\mathcal{C}, \mathrm{Proj}\,\mathcal{C}) = \mathrm{Proj}\,\mathcal{C}$.
   - $\widetilde{\mathcal{T}}^+ = \mathrm{Cone}_{\mathbb{E}}(\mathrm{Inj}\,\mathcal{C}, \mathrm{Inj}\,\mathcal{C}) = \mathrm{Inj}\,\mathcal{C} = \mathrm{Proj}\,\mathcal{C}$.
4. (Nak'18) For any concentric twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$, Hovey $\Rightarrow$ (MT4) holds.

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4. (Nak'18) For any concentric twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$, Hovey $\Rightarrow$ (MT4) holds.

Theorem 2

Let $(\mathcal{S}, \mathcal{Z}, \mathcal{V})$ be a MT satisfying (MT4).
Then Σ is an equiv., in particular, $(\underline{\mathcal{Z}}, \Sigma, \Omega, \nabla, \Delta)$ is a tri. cat.

Sketch of proof

We only prove $\mathrm{Id} \overset{\sim}{\Rightarrow} \Sigma\Omega$.

- From (ET4) and (MT4), we obtain the following $\mathfrak{s}^{\mathcal{I}}$-tri.

$$Z\langle -1\rangle \xrightarrow{{i'}^Z} {I'}^Z \xrightarrow{{p'}^Z} \Psi(Z) \overset{{\lambda'}^Z}{\dashrightarrow} Z\langle -1\rangle$$

$$(\Omega Z)\langle 1\rangle \xrightarrow{\hbar^{(\Omega Z)\langle 1\rangle}} \Psi(Z) \to S \dashrightarrow (\Omega Z)\langle 1\rangle$$

where ${I'}^Z \in \mathcal{I}$, $\Psi(Z) \in \mathcal{Z}$ and $S \in \mathcal{S}$.

- Ψ induces a functor $\Psi\colon \underline{\mathcal{Z}} \to \underline{\mathcal{Z}}$.
- Natural iso. $\psi\colon \mathrm{Id} \overset{\sim}{\Rightarrow} \Psi$ and $\varphi\colon \Psi \overset{\sim}{\Rightarrow} \Sigma\Omega$ are defined as follows.

$$\begin{array}{ccccccc}
Z\langle -1\rangle & \xrightarrow{i_Z} & I_Z & \xrightarrow{p_Z} & Z & \overset{\lambda_Z}{\dashrightarrow} & Z\langle -1\rangle \\
\| & \circlearrowright & \downarrow & \circlearrowright & \downarrow{\scriptstyle \psi_Z} & \circlearrowright & \| \\
Z\langle -1\rangle & \xrightarrow{{i'}^Z} & {I'}^Z & \xrightarrow{{p'}^Z} & \Psi(Z) & \overset{{\lambda'}^Z}{\dashrightarrow} & Z\langle -1\rangle
\end{array}
\qquad
\begin{array}{ccc}
(\Omega Z)\langle 1\rangle & \xrightarrow{\hbar^{(\Omega Z)\langle 1\rangle}} & \Psi(Z) \\
\| & \circlearrowright & \downarrow{\scriptstyle \varphi_Z} \\
(\Omega Z)\langle 1\rangle & \xrightarrow{h^{(\Omega Z)\langle 1\rangle}} & \Sigma\Omega Z
\end{array}$$