Dualizable Grothendieck categories and idempotent rings

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Dualizability

Cocontinuous functors

$\mathcal{C}, \mathcal{D}$: locally presentable additive category (e.g. $\operatorname{Mod} R$, $\operatorname{QCoh} X$)

Definition

- A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called **cocontinuous** if it preserves all (small) colimits.
  (By the special adjoint functor theorem, F is cocontinuous if and only if **it has a right adjoint**.)
- $\operatorname{Cocont}(\mathcal{C}, \mathcal{D}) := \{ \text{cocontinuous functors } \mathcal{C} \to \mathcal{D} \}$.
- $\mathcal{C}^* := \operatorname{Cocont}(\mathcal{C}, \operatorname{Mod} \mathbb{Z})$.

Example (Eilenberg-Watts)

Let R, T be rings.

- $\operatorname{Cocont}(\operatorname{Mod} R, \operatorname{Mod} T) \cong \{ R\text{-}T\text{-bimodules} \} = \operatorname{Mod}(R^{\mathrm{op}} \otimes_{\mathbb{Z}} T)$.
- $(\operatorname{Mod} R)^* = \operatorname{Cocont}(\operatorname{Mod} R, \operatorname{Mod} \mathbb{Z}) \cong \operatorname{Mod}(R^{\mathrm{op}} \otimes_{\mathbb{Z}} \mathbb{Z}) = \operatorname{Mod}(R^{\mathrm{op}})$.

Dualizable categories

$\mathcal{C}, \mathcal{D}$: locally presentable additive category

Definition

$\mathcal{C}$ is **dualizable** if $\mathcal{C}^* \boxtimes \mathcal{C} \xrightarrow{\sim} \mathrm{Cocont}(\mathcal{C}, \mathcal{C})$.

Fact

- $\mathcal{C}$ is dualizable if and only if $\mathcal{C}^* \boxtimes \mathcal{D} \xrightarrow{\sim} \mathrm{Cocont}(\mathcal{C}, \mathcal{D})$ for all $\mathcal{D}$.
- If $\mathcal{C}$ is dualizable, then $\mathcal{C} \xrightarrow{\sim} (\mathcal{C}^*)^*$ (reflexive).

Compare

If V is a finite-dimensional vector space, then $V^* \otimes V \xrightarrow{\sim} \mathrm{Hom}(V, V)$, where V^* is the dual vector space.

Example (Module categories are dualizable)

$(\mathrm{Mod}\, R)^* \boxtimes (\mathrm{Mod}\, R) \cong \mathrm{Mod}(R^{\mathrm{op}}) \boxtimes (\mathrm{Mod}\, R) \cong \mathrm{Mod}(R^{\mathrm{op}} \otimes_{\mathbb{Z}} R)$
$\cong \mathrm{Cocont}(\mathrm{Mod}\, R, \mathrm{Mod}\, R)$.

Conjecture

Proposition (Functor categories are dualizable)

For every small $\mathbb{Z}$-linear category $\mathcal{I}$, the **functor category** of additive functors $\mathcal{I} \to \operatorname{Mod} \mathbb{Z}$ is dualizable.

$\mathcal{C}$: locally presentable additive category

Conjecture (Brandenburg, Chirvasitu, and Johnson-Freyd, 2015)

$\mathcal{C}$ is dualizable if and only if $\mathcal{C}$ is equivalent to a functor category.

Theorem (BCJF 2015)

The conjecture holds for the category of right comodules over a coassociative coalgebra C.(dualizable $\iff$ C is right semiperfect)

Theorem (BCJF 2015)

The conjecture holds for $\operatorname{QCoh} X$ when X is a scheme that has a closed projective subscheme of positive dimension. (never dualizable)

Observation (Roos 1966 + BCJF 2015; Stefanich 2023)

- Jan-Erik Roos investigated a class of Grothendieck categories (containing all functor categories), which we call **Roos categories**.
- BCJF 2015: **Roos categories are dualizable**.
- Roos 1966 or Stefanich 2023: There is a nonzero Roos category that has no nonzero projectives (hence not a functor category).

Modified Conjecture

$\mathcal{C}$ is dualizable if and only if $\mathcal{C}$ is a Roos category.

Theorem (Stefanich 2023 + Kanda 2024)

The modified conjecture is true.

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Roos Categories

Theorem (Gabriel-Popescu 1964)

Let $\mathcal{C}$ be a Grothendieck cat. Fix a generator $U \in \mathcal{C}$ and $R := \mathrm{End}(U)$.

- $\mathrm{Hom}(U, -) \colon \mathcal{C} \to \mathrm{Mod}\, R$ is fully faithful and has a left adjoint Q.
- $Q \colon \mathrm{Mod}\, R \to \mathcal{C}$ is exact, (cocontinuous,) and essentially surjective.
- $\mathcal{X} := \mathrm{Ker}\, Q \subset \mathrm{Mod}\, R$ is a localizing subcat and $\mathcal{C} \cong (\mathrm{Mod}\, R)/\mathcal{X}$.

Theorem (Roos 1965)

In the above setting, TFAE:

1. Q has left adjoint (that is, Q preserves direct products).
2. $\mathcal{X}$ is closed under direct products ($\mathcal{X}$ is bilocalizing).
3. $\mathcal{C}$ satisfies Grothendieck's Ab6 & Ab4* ($\Leftrightarrow$: $\mathcal{C}$ is a **Roos category**).

Remark

- Ab4* $:\Longleftrightarrow$ direct products are exact $\Longleftarrow$ enough projectives
- Ab6 $\Longleftarrow$ functor category or locally noetherian

Observation

Let $\mathcal{C}$ be a Roos category.

- $\mathcal{C} \cong (\operatorname{Mod} R)/\mathcal{X}$ for some bilocalizing $\mathcal{X} \subset \operatorname{Mod} R$.
- **Gabriel 1962:** Every bilocalizing $\mathcal{X}$ is of the form
$$\mathcal{X} = \operatorname{Mod}(R/I) = \{ M \in \operatorname{Mod} R \mid MI = 0 \}$$
for some ideal $I \subset R$ such that $I^2 = I$.
- **Quillen 1996:** $\dfrac{\operatorname{Mod} R}{\operatorname{Mod}(R/I)}$ is equivalent to
$$\operatorname{Mod} I := \{ \text{(non-unital) } I\text{-module } M \text{ such that } M \otimes_I I \xrightarrow{\simeq} M \}.$$
where I is regarded as a **non-unital ring**.

Corollary

For a Grothendieck category $\mathcal{C}$, TFAE:

- $\mathcal{C}$ is a Roos category.
- $\mathcal{C} \cong \dfrac{\operatorname{Mod} R}{\operatorname{Mod}(R/I)}$ for some ring R and $I \subset R$ with $I^2 = I$.
- $\mathcal{C} \cong \operatorname{Mod} I$ for some non-unital ring I with $I^2 = I$.

Modified Conjecture

$\mathcal{C}$ is dualizable if and only if $\mathcal{C}$ is a Roos category.

The modified conjecture follows from the following two theorems:

Theorem (Stefanich 2023 on arXiv)

If $\mathcal{C}$ is dualizable, then $\mathcal{C}$ is a Grothendieck category satisfying Ab4* and $\mathcal{C}^*$ is also a Grothendieck category.

Theorem (Kanda 2024 on arXiv)

If $\mathcal{C}$ is a dualizable and both $\mathcal{C}, \mathcal{C}^*$ are Grothendieck categories, then $\mathcal{C}$ is a Roos category (Ab6 & Ab4*).

Disclaimer: There is a possibility that Stefanich's argument (which I'm not going to talk about) also implies Ab6. In that case, my argument will be an alternative proof (but I believe it's still worthwhile).
In the next slide, I will give a sketch of the proof of my theorem.

Theorem (Kanda 2024 on arXiv)

If $\mathcal{C}$ is a dualizable and both $\mathcal{C}, \mathcal{C}^*$ are Grothendieck categories, then $\mathcal{C}$ is a Roos category (Ab6 & Ab4*).

Sketch of the Proof

- Gabriel-Popescu. $Q\colon \operatorname{Mod} R \to \mathcal{C}$, $S := \operatorname{Hom}(U, -)\colon \mathcal{C} \to \operatorname{Mod} R$.
  Since $\mathcal{C}$ is dualizable, we have:

$$\begin{array}{ccc} \mathcal{C}^* \boxtimes (\operatorname{Mod} R) & \xrightarrow{\sim} & \operatorname{Cocont}(\mathcal{C}, \operatorname{Mod} R) \\ {\scriptstyle \mathrm{id} \boxtimes Q}\downarrow & & \downarrow{\scriptstyle Q_*} \\ \mathcal{C}^* \boxtimes \mathcal{C} & \xrightarrow{\sim} & \operatorname{Cocont}(\mathcal{C}, \mathcal{C}) \ni \mathrm{id}_{\mathcal{C}} \end{array}$$

  Q is ess surj $\rightsquigarrow$ $\mathrm{id} \boxtimes Q$ is ess surj $\rightsquigarrow$ Q_* is ess surj
  $\rightsquigarrow \exists\ H \in \operatorname{Cocont}(\mathcal{C}, \operatorname{Mod} R)$ such that $Q \circ H \cong \mathrm{id}_{\mathcal{C}}$.
- Now $H \circ Q \in \operatorname{Cocont}(\operatorname{Mod} R, \operatorname{Mod} R)$.
  By Eilenberg-Watts, $H \circ Q \cong - \otimes_R B$ for some **bimodule** ${}_R B_R$.
- Let I be the image of the **bimodule map**

$$B \to SQ(B) = SQ(R \otimes_R B) \cong SQHQ(R) \cong SQ(R) \cong R.$$

  Then $I^2 = I$ and $\mathcal{C} \cong \dfrac{\operatorname{Mod} R}{\operatorname{Mod}(R/I)}$, hence a Roos cat.

Consequences

X : noetherian divisorial scheme (e.g. quasi-proj over comm noeth ring).

Theorem (Kanda 2019)

$\operatorname{QCoh} X$ satisifes Ab4* if and only if X is an affine scheme.

Corollary (Generalization of BCJF)

$\operatorname{QCoh} X$ is dualizable if and only if X is an affine scheme.

Summary

For a locally presentable additive category $\mathcal{C}$, TFAE:

1. $\mathcal{C}$ is a Roos category (Grothendieck cat with Ab6 & Ab4*).
2. $\mathcal{C} \cong \dfrac{\operatorname{Mod} R}{\operatorname{Mod}(R/I)}$ for some ring R and $I \subset R$ with $I^2 = I$.
3. $\mathcal{C} \cong \operatorname{Mod} I$ for some non-unital ring I with $I^2 = I$.
4. $\mathcal{C}$ is dualizable.