Cross-connection representations of the category of fibers of vector bundles

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Abstract

- Cross-connectios of normal categories were introduced by K.S.S. Nambooripad in [4]. Let $\mathcal{C}$ and $\mathcal{D}$ be two normal categories, then there exists functor categories $N^*\mathcal{C}$ and $N^*\mathcal{D}$ called the normal duals of $\mathcal{C}$ and $\mathcal{D}$ respectively.
- Then the qudratuple $(\mathcal{C}, \mathcal{D}, \Gamma, \Delta)$ where $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ and $\Delta : \mathcal{C} \to N^*\mathcal{D}$ are certain local isomorphisms is termed as a cross-connection.
- When the normal categories $\mathcal{C}$ and $\mathcal{D}$ are the left and right ideals of a regular semigroup S, then the cross-connection $(\mathcal{C}, \mathcal{D}, \Gamma, \Delta)$ yeilds a cross-connection semigroups which we call the cross connection representation of S.

- V be a vector space over a field K. The singular linear transformation on V written as $Sing(V)$ is a multiplicative semigroup and is one of the most important regular subsemigroups of the regular monoid $\mathcal{T}V$ of all linear transformations on V.
- The semigroup $\mathcal{T}V$ (also called the full linear transformation semigroup) has been an extensively studied semigroup as it is a generalization of the semigroup of matrices, semigroup of operator algebras etc.
- The category of principal left and right ideals of $Sing(V)$ are normal categories.

- These ideal categories of $Sing(V)$ are characterized as the subspace category $\mathcal{S}(V)$ of proper subspaces of V with linear transformations as morphisms and the annihilator category $\mathcal{A}(V)$ which is isomorphic to the category $\mathcal{S}(V^*)$ of proper subspaces of V^* where V^* is the algebraic dual space of V.
- These ideal categories of $Sing(V)$ and their normal duals yeilds a cross-connection $(\mathcal{C}, \mathcal{D}, \Gamma)$ since the local isomorphism Δ can conveniently replaced by Γ^*. This yeilds a a cross-connection representation for the semigroup $Sing(V)$ (cf.[6]).

- A Lie groupoid is a groupoid with additional smooth manifold structures on its objects and the morphisms that makes various maps arise from the groupoid structure smooth.
- A bundle is a triple $\eta = (E, p, B)$ is where $p : E \rightarrow B$ is a projection, B is the base space and E the total space. A bundle in which each fibre $p^{-1}(b)$ admits a vector space structure is a vector bundle.
- In this talk we extend cross-connection representation to the category of fibers of a vector bundle

This talk is divided into six sections as follows

- Categories, preorders and normal categories
- Cross-connections of normal categories
- Cross-connection semigroup of linear transformations
- Groupoids, Lie groupoids and Lie categories
- Bundles and vector bundles
- Category of fibers of a vector bundle
- Amalgamated products and cross-connection semigroup of bundles

A category $\mathcal{C}$ consists of the following data:

1. A class called the class of vertices or objects $\nu\mathcal{C}$.
2. A class of disjoint sets $\mathcal{C}(a, b)$ one for each pair $(a, b) \in \nu\mathcal{C} \times \nu\mathcal{C}$. An element $f \in \mathcal{C}$ is called a morphism from a to b, written $f : a \to b$; $a = dom\ f$ the domain of f and $b = cod\ f$ called the codomain of f.
3. For $a, b, c \in \nu\mathcal{C}$, a map

$$\circ : \mathcal{C}(a, b) \times \mathcal{C}(b, c) \to \mathcal{C}(a, c) \text{ given by } (f, g) \mapsto f \circ g$$

   is the composition of morphisms in $\mathcal{C}$.
4. for each $a \in \nu\mathcal{C}$, a unique $1_a \in \mathcal{C}(a, a)$ is the identity morphism on a.

(cont...)

These must satisfy the following axioms:

- for $f \in \mathcal{C}(a, b)$,
$$g \in \mathcal{C}(b, c) \textit{ and } h \in \mathcal{C}(c, d),$$
then
$$f \circ (g \circ h) = (f \circ g) \circ h$$
- for each $a \in \nu\mathcal{C}, f \in \mathcal{C}\ (a, b)$ *and* $g \in \mathcal{C}(c, a)$, then
$1_a \circ f = f \qquad \textit{and} \qquad g \circ 1_a = g.$

Examples

- **Set**: the category in which objects are sets and morphisms are functions between sets.
- **Grp**: Category with groups as objects and homomorphisms as morphisms.
- **Groups**: A category with one object namely the **Group** whose morphisms (arrows) are the elements of the **Group**. Hence every arrow has a (two sided) inverse under composition.

Functors

A **functor** $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of a vertex map $\nu F : \nu\mathcal{C} \to \nu\mathcal{D}$ which assigns to each $a \in \nu C$ a vertex $F(a) \in D$ and a morphism map $F : C \to D$ which assigns to each morphism $f : a \to b$, a morphism

$$F(f) : F(a) \to F(b) \in \mathcal{D}$$

such that

1. $F(1_a) = 1_{F(a)}$ for all $a \in \nu\mathcal{C}$; and
2. $F(f)F(g) = F(fg)$ for all morphisms $f, g \in \mathcal{C}$ for which the composition fg exists.

Example

The power set functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$. Its object function assigns each object X in $\mathbf{Set}$ the usual power set $\mathcal{P}X$ and its arrow function assigns to each $f : X \to Y$ the map $\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y$ which send each $S \subset X$ to its image $fS \subset Y$.

Natural transformation

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A **natural transformation** $\eta : F \to G$ is a family $\{\eta_a : F(a) \to G(a) | a \in \nu\mathcal{C}\}$ of maps in $\mathcal{D}$ such that for every map $f : a \to b$ in $\mathcal{C}$, the following diagram commuts

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta_c} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow[\eta_{c'}]{} & G(b) \end{array}$$

The map η_a are called the components of η. If each component of η is an isomorphism then η is called a natural isomorphism.

Functor categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $S, T \ldots$ be functors from $\mathcal{C} \to \mathcal{D}$. A category whose oblects are functors between categories and morphisms are natural transformations between such functors with composition of morphisms, the composition of natural transformations is a category and is termed as the functor category.

Preorder

A preorder $\mathcal{P}$ is a category such that for any $p, p' \in v\mathcal{P}$, the hom-set $\mathcal{P}(p, p')$ contains at most one morphism.
In this case, the relation $\subseteq$ on the class $v\mathcal{P}$ of objects of $\mathcal{P}$ is defined by $p \subseteq p'$ if $\mathcal{P}(p, p') \neq \emptyset$ is a quasi- order.
$\mathcal{P}$ is said to be a strict preorder if $\subseteq$ is a partial order.

Definition

(Category with subobjects) Let $\mathcal{C}$ be a small category and $\mathcal{P}$ be a subcategory of $\mathcal{C}$ such that $\mathcal{P}$ is a strict preorder with $v\mathcal{P} = v\mathcal{C}$. Then $(\mathcal{C}, \mathcal{P})$ is a category with sub objects if

1. *every $f \in \mathcal{P}$ is a monomorphism in $\mathcal{C}$*
2. *if $f = hg$ for $f, g \in \mathcal{P}$, then $h \in \mathcal{P}$.*

Example

In categories Set, Grp, $Vect_K$, Mod_R the relation on objects induced by the usual set inclusion is a subobject relation.

Factorization of morphisms

In a category $(\mathcal{C}, \mathcal{P})$ with subobjects, morphisms in $\mathcal{P}$ are called inclusions. If $c' \to c$ is an inclusion, we write $c' \subseteq c$ and denotes this inclusion by $j_{c'}^{c}$. An inclusion $j_{c'}^{c}$ splits if there exists $q : c \to c' \in \mathcal{C}$ such that $j_{c'}^{c} q = 1_{c'}$ and the morphism q is called a retraction.

Definition

A morphism f in a category $\mathcal{C}$ with subobjects is said to have factorization if f can be expressed as $f = pm$ where p is an epimorphism and m is an embedding.

A normal factorization of a morphism $f \in \mathcal{C}(c, d)$ is a factorization of the form $f = quj$ where $q : c \to c'$ is a retraction, $u : c' \to d'$ is an isomorphism and $j = j_{d'}^{d}$ is an inclusion where $c', d' \in v\mathcal{C}$ with $c' \subseteq c, \ d' \subseteq d$. The morphism qu is known as the epimorphic component of f and is denoted by f°.

Cones in a category

Definition

Let $\mathcal{C}$ be a category with subobjects and $d \in v\mathcal{C}$. A map $\gamma : v\mathcal{C} \to \mathcal{C}$ is called a cone from the base $v\mathcal{C}$ to the vertex d if

1. $\gamma(c) \in \mathcal{C}(c, d)$ *for all* $c \in v\mathcal{C}$
2. *if* $c \subseteq c'$ *then* $j_c^{c'}\gamma(c') = \gamma(c)$

For a cone γ denote by c_γ the vertex of γ and for $c \in v\mathcal{C}$, the morphism $\gamma(c) : c \to c_\gamma$ is called the component of γ at c. A cone γ is said to be normal if there exists $c \in v\mathcal{C}$ such that $\gamma(c) : c \to c_\gamma$ is an isomorphism. We denote by $T\mathcal{C}$ the set of all normal cones in $\mathcal{C}$

Definition

A category $\mathcal{C}$ with subobjects is called a normal category if the following holds

1. *any morphism in $\mathcal{C}$ has a normal factorization*
2. *every inclusion in $\mathcal{C}$ splits*
3. *for each $c \in v\mathcal{C}$ there is a normal cone γ with vertex c and $\gamma(c) = 1_{c_\gamma}$.*

Observe that given a normal cone γ and an epimorphism $f : c_\gamma \to d$ the map $\gamma * f : a \to \gamma(a)f$ from $v\mathcal{C}$ to $\mathcal{C}$ is a normal cone with vertex d.

Normal dual

Remark

The set of all normal cones $T\mathcal{C}$ in $\mathcal{C}$ with the cone composition

$$\gamma^1 \cdot \gamma^2 = \gamma^1 * (\gamma^2_{c_{\gamma^1}})^\circ$$

is a regular semigroup.

For each $\gamma \in T\mathcal{C}$, $H(\gamma, -) : \mathcal{C} \to \mathbf{Set}$ is a set values functor. A category whose objects are such set valued functors together with natural tranformations between such functors as morphisms is a functor category and is termed as the normal dual of $\mathcal{C}$, denoted by $N^*\mathcal{C}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be normal categories, $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ and $\Delta : \mathcal{C} \to N^*\mathcal{D}$ be local isomorphisms. Then Γ and Δ induces bi- functors $\Gamma(-,-) : \mathcal{D} \times \mathcal{C} \to \mathbf{Set}$ and $\Delta(-,-) : \mathcal{C} \times \mathcal{D} \to \mathbf{Set}$ such that there is a natural bijection

$$\chi : \Gamma(-,-) \to \Delta(-,-).$$

Then $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ and $\Delta : \mathcal{C} \to N^*\mathcal{D}$ are termed as a connection and dual connection between the categories $\mathcal{C}$ and $\mathcal{D}$

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be normal categories. A cross-connection is a triple $(\mathcal{D}, \mathcal{C}, \Gamma)$ where $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ is a local isomorphism such that for every $c \in \nu\mathcal{C}$ there is some $d \in \nu\mathcal{D}$ with $c \in M\Gamma(d)$.

Note that for a cross-connection $(\mathcal{D}, \mathcal{C}, \Gamma)$ there corresponds a dual cross-connection $(\mathcal{C}, \mathcal{D}, \Gamma^*)$

The cross-connection semigroup

Let $\mathcal{C}$ and $\mathcal{D}$ be normal categories such that $(\mathcal{D}, \mathcal{C}, \Gamma)$ is a cross-connection. Then for each $(c, d) \in \mathcal{C} \times \mathcal{D}$ there is a natural bijection

$$\chi_{\Gamma(c,d)} : \Gamma(c, d) \to \Gamma^*(c, d)$$

between the bi-functors $\Gamma : \mathcal{D} \times \mathcal{C} \to \mathbf{Set}$ and $\Gamma^* : \mathcal{C} \times \mathcal{D} \to \mathbf{Set}$ such that this bijection yeilds a pairs of cones (γ, γ^*) and the collection of these cones together with the binary composition defined by

$$(\gamma, \gamma^*) \circ (\delta, \delta^*) = (\gamma\delta, \delta^*\gamma^*)$$

is a semigroup and is called the corss-connection semigroup which is denoted by $\tilde{S}\Gamma$.

Cross-connection semigroup of singular endomorphisms $Sing(V)$ on V

It is well known that the singular endomorphisms on a vector space V written as $Sing(V)$ is a multiplicative semigroup.
The following properties of ideal categories of $Sing(V)$ can be observed (see cf.[6])

1. The category of principal left ideals of $Sing(V)$ is characterized as the subspace category $\mathcal{S}(V)$ of proper subspaces of V with linear transformations as morphisms.
2. The category $\mathcal{S}(V)$ is a normal category and its set of all normal cones under the binary operation of cone composition forms the regular semigroup $\mathcal{TS}(V)$.
3. The semigroup of all normal cones in the category of principal left ideals of $Sing(V)$ is isomorphic to $Sing(V)$.

1. The normal dual of the category $\mathcal{S}(V)$ is the annihilator category $\mathcal{A}(V)$ which is isomorphic to the category $\mathcal{S}(V^*)$ of proper subspaces of V^* where V^* is the algebraic dual space of V

.

Proposition

(Proposition 5. cf.[6]) The normal dual $N^*\mathcal{S}(V)$ of the subspace category is isomorphic to $\mathcal{A}(V)$. Dually $N^*\mathcal{A}(V)$ is isomorphic to $\mathcal{S}(V)$. The semigroup $T\mathcal{A}(V)$ of normal cones in $\mathcal{A}(V)$ is isomorphic to $Sing(V)^{op}$.

Theorem

(Theorem 6. cf.[6]) Let V be a finite diamensional vector space over K, then $N^*\mathcal{S}(V)$ is isomorphic as a normal category to $\mathcal{S}(V^*)$

Now it is easy to observe that there exists local isomorphisms

$$\Gamma : \mathcal{A}(V) \to \mathcal{S}(V^*) \quad \text{and}$$
$$\Gamma^* : \mathcal{S}(V) \to \mathcal{A}(V^*),$$

and for $\Gamma : \mathcal{A}(V) \to \mathcal{S}(V^*)$ there is a unique bifunctor $\Gamma(-,-) : \mathcal{A}(V) \times \mathcal{S}(V^*) \to \mathbf{Set}$ such that for all $(A, Y) \in \mathcal{A}(V) \times \mathcal{S}(V^*)$ and $(f, w^*) : (A, Y) \to (B, Z)$

$$\Gamma(A, Y) = \{\alpha \in \mathcal{T}V : V\alpha \subseteq A \,\text{and}\, (A\alpha)^\circ \subseteq \Gamma(Y)\}$$
$$\Gamma(f, w^*) : \alpha \mapsto (y\alpha)f = y((\alpha f) \quad \text{where } y \text{ is given by } y^* = \Gamma(w^*)$$

Similarly we can have the bifunctor $\Gamma^*(-,-)$. Moreover these bifunctors $\Gamma(-,-)$ and $\Gamma^*(-,-)$ are connected by a natural bijection χ_Γ given by

$$\chi_{\Gamma_\epsilon}(A, Y) : \alpha \mapsto \epsilon^{-1}\alpha\epsilon$$

and the natural bijection χ_{Γ_ϵ} provides a pair of cones (α, β) where

$$\beta = \chi_{\Gamma_\epsilon}(\alpha) = \epsilon^{-1}\alpha\epsilon.$$

The cross-connection semigroup associated with this cross-connections is given by

$$\tilde{S}\Gamma_\epsilon = \{(\alpha, \epsilon^{-1}\alpha\epsilon) \quad \text{such that}\, \alpha \in Sing(V)\}.$$

Further we have the following theorem

Theorem

(cf. [6] Theorem 8) Every cross-connection semigroup arising from the cross-connections between the categories $\mathcal{A}(V)$ and $\mathcal{S}(V)$ is isomorphic to $Sing(V)$.

Groupoids

Definition

A groupoid consists of two sets G and M, called respectively the groupoid and the base, together with two maps α and β from G to M, called the source projection and target projection, a map $1 : x \mapsto 1_x$ the object inclusion map and a partial multiplication in G defined on the set $G * G = \{(h, g) \in G \times G \,|\, \alpha(h) = \beta(g)\}$, subject to the following conditions:

1. $\alpha(hg) = \alpha(g)$ *and* $\beta(hg) = \beta(h)$ *for all* $(h, g) \in G * G$:
2. $j(hg) = (jh)g$ *for all* $j, h, g \in G$ *such that* $\alpha(j) = \beta(h)$, $\alpha(h) = \beta(g)$;
3. $\alpha(1_x) = \beta(1_x) = x$ *for all* $x \in M$;
4. $g1_{\alpha(g)} = g$ *and* $(1_{\beta(g)}g = g$ *for all* $g \in G$;
5. *each* $g \in G$ *has a two sided inverse* g^{-1} *such that* $\alpha(g^{-1}) = \beta(g), \beta(g^{-1}) = \alpha(g)$ *and* $g^{-1}g = 1_{\alpha(g)}, gg^{-1} = 1_{\beta(g)}$.

Here $G * G = (\alpha \times \beta)^{-1}(\Delta_M)$ *is a closed embedded submanifold of* $G \times G$, *since* α *and* β *are submersions.*

Element of M are called objects of the groupoid G and elements of G are called arrows. The arrow 1_x corresponds to the object $x \in M$ may also be called the unity or identity corresponding to x.

Example

- *Obviously every Group is a groupoid*
- *Fundamental groupoid of a space*

Definition

A Lie groupoid $G \rightrightarrows M$ is a groupoid G on base M together with smooth structures on G and M such that the maps $\alpha, \beta : G \to M$ are surjective submersions, the object inclusion map $x \mapsto 1_x$, $M \to G$ is smooth and the partial multiplication $G * G \to G$ is smooth.

Here $G * G = (\alpha \times \beta)^{-1}(\Delta_M)$ is a closed embedded submanifold of $G \times G$, since α and β are submersions.

Example

Let $\mathbf{G} \times \mathbf{M} \to \mathbf{M}$ *be a smooth action of a Lie group* $\mathbf{G}$ *on a manifold* $\mathbf{M}$*. On the product manifold* $\mathbf{G} \times \mathbf{M}$ *we can give the structure of a Lie groupoid in the following way.*

1. α *be the projection into the second factor of* $\mathbf{G} \times \mathbf{M}$, β *be the group action itself.*
2. *The object inclusion map is* $x \mapsto 1_x = (1, x)$.
3. *partial multiplication is* $(g_2, y)\,(g_1, x) = (g_2 g_1, x)$ *which is defined if* $y = g_1 x$.
4. *The inverse of* (g, x) *is* (g^{-1}, gx).

with this structure we denote $\mathbf{G} \times \mathbf{M}$ *by* $\mathbf{G} \lessdot \mathbf{M}$ *and call it the action groupoid of* $\mathbf{G} \times \mathbf{M} \to \mathbf{M}$.

Example

Any manifold **M** *may be regarded as a Lie groupoid on itself with* $\alpha = \beta = id_{\mathbf{M}}$ *and every element a unity.*

A groupoid in which every element is a unity will be called a base groupoid.

Given any Lie groupoid $G \rightrightarrows M$*, the units of* M *define a subgroupoid*

$$\begin{array}{ccc} M & \rightrightarrows & M \\ \downarrow & & \downarrow \\ G & \rightrightarrows & M \end{array}$$

Example

Let $\mathbf{M}$ *be a manifold and* $\mathbf{G}$ *a Lie group. We give* $\mathbf{M} \times \mathbf{G} \times \mathbf{M}$ *the structure of a Lie groupoid on* $\mathbf{M}$ *in the following way:*

- α *is the projection into the third factor of* $\mathbf{M} \times \mathbf{G} \times \mathbf{M}$ *and* β *is the projection into the first factor.*
- *The object inclusion map is* $x \mapsto 1_x = (x, 1, x)$.
- *partial multiplication is* $(z, h, y')\ (y, g, x) = (z, hg, x)$ *which is defines iff* $y = y'$.
- *The inverse of* (y, g, x) *is* (x, g^{-1}, y).

this is usually called the trivial groupoid on $\mathbf{M}$ *with group* $\mathbf{G}$*. In particular any Lie group may considered to be a Lie groupoid on a singleton manifold and any cartesian square* $\mathbf{M} \times \mathbf{M}$ *is a groupoid on* $\mathbf{M}$ *which is the pair groupoid*

$$Pair(M) = \mathbf{M} \times \mathbf{M} \rightrightarrows M.$$

Lie category

Definition

A Lie category is a small category $\mathcal{C} \rightrightarrows M$, where $\mathcal{C}$ is a smooth manifold with or without boundary, M is a smooth manifold without boundary, and there holds:

1. *The source and target maps $s, t : \mathcal{C} \to M$ are smooth submersions.*
2. *The unit map $u : M \to \mathcal{C}$ and the composition map $m : \mathcal{C}^2 \to \mathcal{C}$ are smooth. If $\mathcal{C}$ has a boundary, we also assume that $\mathcal{C} \rightrightarrows M$ has a regular boundary, that is:*
3. *The restrictions $\partial s, \partial t : \partial\mathcal{C} \to M$ of s and t are smooth submersions.*

Remark

Clearly every Lie groupoid is a Lie category, but the reverse is not always true.

A *fiber bundle* (simply a bundle) is a triple $\eta = (E, p, B)$ is where $p : E \to B$ is a map. The space B is called the base space and the space E is called total space and the map p is called the projection of the bundle. For each $b \in B$, the space $p^{-1}(b)$ is called the *fibre* of the bundle over $b \in B$.
Intuitively a bundle is regarded as a union of fibres $p^{-1}(b)$ for $b \in B$ parametrized by B and 'glued together' by the topology of the space E.

Example

A covering space is a(continuous, surjective) map $p : X \to Y$ such that for every $y \in Y$ there exist an open neighborhood U containing y such that $p^{-1}(U)$ is homeomorphic to a disjoint union of open sets in X, each being mapped homeomorphically onto U by p . Hence it is a fibre bundle such that the bundle projection is a local homeomorphism.

Example

Given any space B, a product bundle over B with fibre F is the bundle $(B \times F, p, B)$ where p is the projection on the first factor.

Definition

A bundle $\eta' = (E', p', B')$ is a subbundle of $\eta = (E, p, B)$ provided E' is a subspace of E, B' is a subspace of B and $p' = p|\, E' : E' \to B'$.

Example

The tangent bundle over S^n denoted as (T, p, S^n) and the normal bundle (N, q, S^n) are two subbundles of the product bundle $(S^n \times R^{n+1}, p, S^n)$ whose total space are defined by the relation $(b, x) \in T$ if and only if the inner product $\langle b, x \rangle = 0$ and by $(b, x) \in N$ if and only if $x = kb$ for some $k \in R$.

Two bundles η and ζ over B are locally isomorphic if for each $b \in B$ there exists an open neighbourhood U of b such that $\eta|U$ and $\zeta|U$ are U-isomorphic.

Definition

Consider the bundle (E, p, B). A map $s : B \to E$ is called a section (also cross section) of the bundle (E, p, B) if $p \circ s = id_B$

A space F is a fibre of a bundle (E, p, B) provided every fibre $p^{-1}(b)$ for $b \in B$ is homoeomorphic to F. A bundle (E, p, B) is trivial with fibre F provided (E, p, B) is B-isomorphic to the product bundle $(E \times F, p, B)$.

Definition

A bundle η over B is locally trivial with fibre F if η is locally isomorphic with the product bundle $(B \times F, p, B)$

A fibre bundle with fibre F is also written as (E, p, B, F), where E, B and F are topological spaces, and $p : E \to B$ is a continous surjection satisfying local trivality condition.

Bundle morphisms

A bundle morphism is a pair of fibre preserving map between two bundles $\eta = (E, p, B)$ and $\eta' = (E', p', B')$ given by

$$(u, f) : (E, p, B) \to (E', p', B')$$

where $u : E \to E'$ and $f : B \to B'$ such that $p'u = fp$, ie., the diagram of arrows commutes.

In particular when $\eta = (E, p, B)$ and $\eta' = (E', p', B)$ bundles over B, then the bundle morphism $u : (E, p, B) \to (E', p', B)$ is a map $u : E \to E'$ such that $p = p'u$.

Example

Let (E', p', B') is a subbundle of (E, p, B) . If $f : B' \to B$ and $u : E' \to E$ are inclusion maps, then $(u, f) : (E', p', B') \to (E, p, B)$ is a bundle morphism.

Category of Bundles (BUN)

Definition

The category of bundles, denoted **BUN***, has as its objects all bundles* (E, p, B) *and as morphisms* (E, p, B) *to* (E', p', B') *the set of all bundle morphisms.*
For each space B*, the sub category of bundles over* B*, denoted as* $\mathbf{BUN_B}$ *has objects bundles with base* B *and* B*-morphisms as its morphisms.*

Vector Bundles

A bundle with an additional vector space structure on each fibre is called a vector bundle.

Definition

A k- dimensional vector bundle over a field F is a bundle (E, p, B) together with the structure of a k- dimensional vector space on each fibre $p^{-1}(b)$ such that the following local triviality condition is satisfied. Each point of B has an open neighborhood U and a U- isomorphism h: $U \times F^k \rightarrow p^{-1}(U)$ such that the restriction $b \times F^k \rightarrow p^{-1}(b)$ is a vectorspace isomorphism for each $b \in U$.

Example

If M is a smooth manifold and k is a nonnegative integer then

$$p : M \times \mathbb{R}^k \rightarrow M$$

is a real vector bundle of rank k over M and is a trivial vector bundle.

Example

Let M be a smooth $n-$ manifold, the collection of tangent vectors TM of M is a vector bundle of rank n over M. As a set it is given by disjoint union of tangent spaces of M. Elements of TM can be thought of as a pair (x, v) where x is a point in M and v is a tangent vector to M at x. There is a natural projection $\pi : TM \to M$ denoted $\pi(x, v) = x$ and the vector bundle is (TM, π, M) where each fibre is the tangent space at some point of M.
The cotangent bundle may describe in a similar way and is dented by T^*M.

Example

The tangent bundle over S^n, denoted (T, p, S^n), and the normal bundle over S^n, denoted (N, q, S^n) are two subbundles of the product bundle $(S^n \times R^{n+1}, p, S^n)$ whose total spaces are defined by the relation $(b, x) \in T$ if and only if the inner product $\langle b, x \rangle = 0$ and by $(b, x) \in N$ if and only if $x = kb$ for some $k \in R$.

Definition

Let $p : V \to M$ is a vector bundle. A section of p or V is a smooth map $s : M \to V$ such that $p \circ s = id_M$ ie., $s(x) \in V_x$ for all $x \in M$.

Note that if s is a section then $s(M)$ is an embedded submanifold of V.

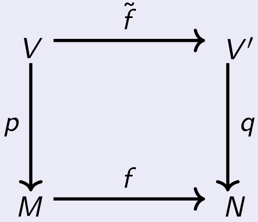
Lemma

Every vector bundle admits a section.

Definition

Suppose $p : V \to M$ and $q : V' \to N$ are two real vector bundles.

- *(i) A smooth map $\tilde{f} : V \to V'$ is a vector bundle morphism if and only if $\tilde{f}$ decends to a map $f : M \to N$ such that $q\tilde{f} = fp$, ie., the diagram below commutes*



and the restriction $\tilde{f} : V_x \to V_{f(x)}$ is linear for all $x \in M$.

- *(ii) If $p : V \to M$ and $q : V' \to M$ are vector bundles, a vector bundle morphism $\tilde{f} : V \to V'$ is an isomorphism of vector bundles if $q\tilde{f} = p$. If such an isomorphism exists then V and V' are said to be isomorphic.*

Frame groupoid of a vector bundle

Definition

Let $\pi : E \to M$ be a vector bundle. $\mathcal{G}(E)$ denotes the category whose object set is M and morphisms are isomorphisms $E_x \to E_y$ where $x, y \in M$ and E_x, E_y are fibres of the vector bundle at x, y respectively. It is easy to observe that $\mathcal{G}(E)$ is a groupoid and is called the frame groupoid of $\pi : E \to M$.

Remark

The frame groupoid of a vector bundle is locally trivial.

Category of vector bundles

The category of vector bundles, denoted by **VB**, has as its objects vector bundles and its morphisms smooth maps which are vector bundle morphisms.

Note that a vector bundle morphism between two vector bundles $\eta = (E, p, B)$ and $\eta' = (E', p', B')$ is a morphism of the underlying bundles.

For a base space B the category $\mathbf{VB_B}$ is a subcategory of vector bundle category over B and B-morphisms.

A remarkable property of a vector bundle $E \to M$ is that it affords a representation of a Lie groupoid $G \rightrightarrows M$ defined as follows

Definition

Let $G \rightrightarrows M$ be a Lie groupoid. A representation of G on a vector bundle $E \to M$ is a smooth homomorphism

$$\Delta : G \to \mathcal{G}(E) \quad \textit{of the Lie groupoid over}$$

where $\mathcal{G}(E)$ is the frame groupoid.

Category of fibers of a vector bundles

Consider the vector bundle $p : V \to M$ with base M of dimension k. Then each fiber $p^{-1}(m)$ of the bundle $p : V \to M$ is a k dimensional vector space V_m. Now it is easy to observe that any subspace of V_m is a fibre of some vector bundle of dimension $" \leq " k$.

Definition

For a vector bundle $p : V \to M$ of dimension k, there is a category termed as the category of fibers of vector bundles whose object is the set of fibers of diamension $" \leq " k$ and smooth linear maps between these objects as morphisms and we denote this category by $\mathcal{C}_{fb}(M)$.

Lemma

Let $p : V \to M$ be a vector bundle of dimension k. Then the category $\mathcal{C}_{fb}(M)$ of fibers of the vector bundle $p : V \to M$ is a Lie category.

Proof.

Since the objects of the category being vector spaces (fibers), they are manifolds with out boundary and the morphisms are linear maps, it is easy to observ that $\mathcal{C}_{fb}(M)$ is a Lie categry whose source and target maps are the domain and range map of the smooth linear transformation. □

CROSS-CONNECTIONS OF CATEGORY OF FIBERS OF A VECTOR BUNDLE

Next we propose to investigate the cross-connections in the category of fibers of a vector bundle $p : V \to M$ of dimension k from the cross-connection representation of the semigroup of singular linear endomorphams $Sing(V_\alpha)$ of the fiber V_α of the bundle $p : V \to M$.

Let $\{S_i : i \in I\}$ be a disjoint union of semigroups. Then there is an $F = \Pi^*\{S_i : i \in I\}$ called the free product of the family $\{S_i : i \in I\}$ satisfying the following Proposition.

Proposition

$F = \Pi^*\{S_i : i \in I\}$ be a product of the family $\{S_i : i \in I\}$ of disjoint semigroups such that for each $i \in I$ there is a monomorphism $\theta_i : S_i \to F$. If T is a semigroup for which a homomorphism $\psi_i : S_i \to T$ exists for each i in I then there is a unique homomorphism $\gamma : F \to T$ such that $\theta_i\gamma = \psi$.

Amalgamated product of semigroups

Let $\{S_i \,:\, i \in I\}$ be a family of semigroups, and suppose that for each $i \in I$ there exists a homomorphism h_i from a semigroup U into S_i. Consider the set $\Sigma = \{(x, i) : i \in I, x \in S_i\}$ the sum of sets S_i and construct the semigroup S presented by

$$S = \langle \Sigma; (x,i)(x',i) = (xx',i) \text{ for every } x \in I, x.x' \in S_i$$
$$(h_i(u), i) = h_j(u), j) \text{ for every } \; i, j \in I, u \in U\rangle$$

then S is called the product of the family S_i amalgamated by U. If ϕ denotes the canonical homomorphism from the free semigroup on the alphabet Σ onto S, then ϕ_i defined by $\phi_i(x) = \phi(x, i)$ is a homomorphism from S_i into S and $h : U \to S$ defined by $h(u) = \phi[h_i(u), i]$ is independent of i. Then there is a family $\{S_i \,:\, i \in I\}$ of semigroups together with injective homomorphisms $h_i : U \to S_i$ for all $i \in I$ and the system $[S_i; U; h_i]_{i \in I}$ is called a semigroup amalgam (cf.[5]).

Definition

A semigroup amalgama $\mathcal{U} = [\{S_i : i \in I\}; U; \{\phi_i : i \in I\}]$ *consists of semigroup* U *called the core of the amalgam, a family* $\{S_i : i \in I\}$ *of semigroups disjoint from each other and from* U*, and a family of monomorphisms* $\phi_i : U \to S_i \, (i \in I)$*.*

The amalgm $\mathcal{U}$ is said to be embeddeble in a semigroup T if there exists a monomorphism $\lambda : U \to T$ and for each $i \in I$ the monomorphiSms $\lambda_i : S_i \to T$ which satisfies

1. $\phi_i \lambda_i = \lambda$ for each i in I
2. $S_i \lambda_i \cap S_j \lambda_j = U\lambda$ for all $i, j \in I$ such that $i \neq j$.

Amalgamated product cross-connection semigroups

Let $p : E \to M$ be a vector bundle whose fibers are vector spaces $\{V_b : b \in B\}$ and the cross-connection semigroup associate to the fiber V_b is

$$\tilde{S}_b = \tilde{S}\Gamma_\epsilon = \{(\alpha, \epsilon^{-1}\alpha\epsilon) \, \text{such that} \, \alpha \in Sing(V_b)\}.$$

The amalgamated product of cross-connection semigroups $\tilde{S}_b$ is described in the following proposition.

Proposition

Let $p : E \to M$ be a vector bundle whose fibers are vector spaces $\{V_b : b \in B\}$ and family of semigroups $\tilde{S}_b = \{(\alpha, \epsilon^{-1}\alpha\epsilon)\, \text{such that}\, \alpha \in Sing(V_b)\}$. Then there is a semigroup U disjoint from the family of semigroups $\tilde{S}_b$ and a family of monomorphisms $\{\phi_i : U \to \tilde{S}_b\}$ such that

$$\mathcal{U} = [\{\tilde{S}_b = (\alpha, \epsilon^{-1}\alpha\epsilon) \,:\, \alpha \in Sing(V_b)\}; U; \{\phi_i :\ i \in I\}]$$

is a semigroup amalgamam.

Problem

Is the semigroup amalgam described in the above proposition embedable? (If embedable then show that it is the corss-connection semigroup of the vector bundle.)

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THANK YOU