Categorifying twisted Auslander-Reiten quivers

ICRA 21 - Shanghai

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August 8, 2024

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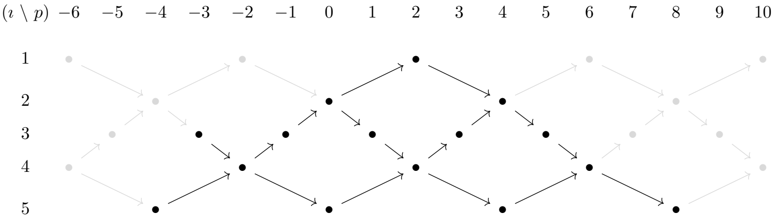
Can we categorify these combinatorics? Is there a "category of representations" for a Q-datum?

Twisted AR quiver: type B_3

Highlighted subquiver: a twisted AR quiver of type B_3.



The vertices are in bijection with the positive roots of A_5.

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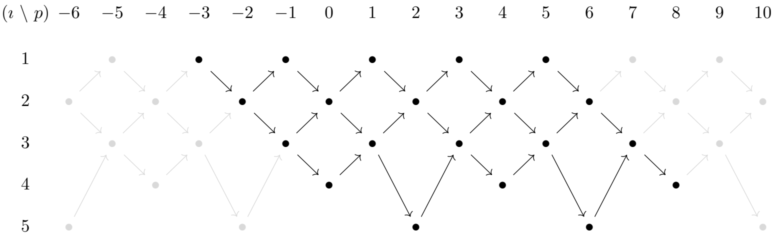

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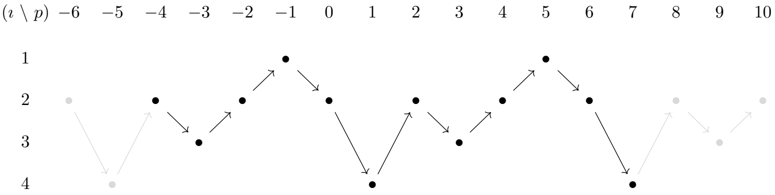

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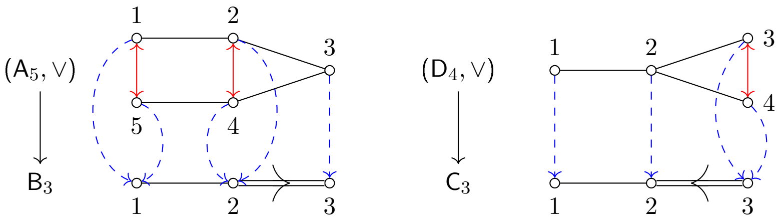
Unfoldings

Let (Δ, σ) be the unfolding of $\mathfrak{g}$, where Δ is a simply-laced Dynkin diagram and σ is an automorphism of Δ.



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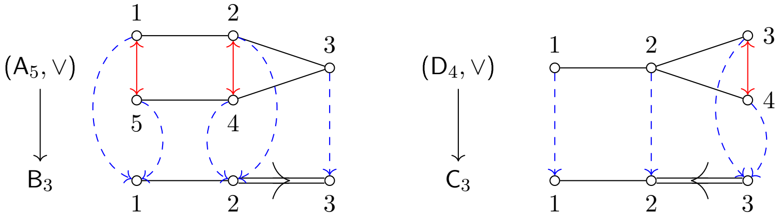

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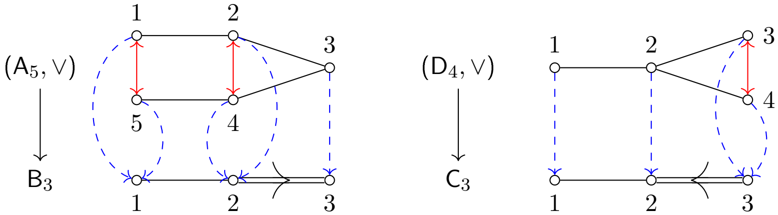

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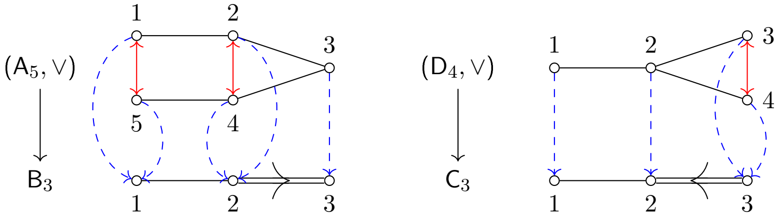

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If $i \in I$, we define $d_i = |i| \in \{1, r\}$, where r is the order of σ.

Q-data

Definition [Fujita-Oh '21]

A Q-datum for $\mathfrak{g}$ is a triple $\mathcal{Q} = (\Delta, \sigma, \xi)$ where (Δ, σ) is the unfolding of $\mathfrak{g}$ and $\xi : \Delta_0 \to \mathbb{Z}$ satisfies:

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- For adjacent $\imath, \jmath \in \Delta_0$ with $d_{\bar{\imath}} = d_{\bar{\jmath}}$, we have $|\xi_\imath - \xi_\jmath| = d_{\bar{\imath}} = d_{\bar{\jmath}}$.
- For adjacent $i, j \in I$ with $d_i = 1 < d_j = r$, there is a unique $\jmath \in j$ such that $|\xi_\imath - \xi_\jmath| = 1$ and $\xi_{\sigma^k(\jmath)} = \xi_\jmath - 2k$ for any $1 \leq k < r$, where $i = \{\imath\}$.

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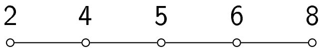
For example, the following height function defines a Q-datum of type B_3:



Repetition quivers and twisted AR quivers

Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum. The repetition quiver $\widehat{\Delta}^{\sigma}$ has vertices:

$$\widehat{\Delta}^{\sigma}_0 = \{(\imath, p) \in \Delta_0 \times \mathbb{Z} \mid p - \xi_\imath \in 2d_{\bar{\imath}}\mathbb{Z}\}.$$

There is an arrow $(\imath, p) \longrightarrow (\jmath, s)$ if $\imath$ and $\jmath$ are adjacent in Δ and $s - p = \min(d_{\bar{\imath}}, d_{\bar{\jmath}})$.

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The twisted Auslander-Reiten quiver $\Gamma_{\mathcal{Q}}$ is the full subquiver of $\widehat{\Delta}^\sigma$ with vertex set

$$\{(\imath, p) \in \widehat{\Delta}^\sigma_0 \mid \xi_{\imath^*} - rh^\vee < p \leq \xi_\imath\},$$

where:

- $h^\vee$: dual Coxeter number of $\mathfrak{g}$.
- $\imath \mapsto \imath^*$: involution on Δ_0 induced by the longest element w_0 of the Weyl group associated with Δ.

Assigning positive roots

A compatible reading of $\Gamma_{\mathcal{Q}}$ is an ordering $(\imath_1, p_1), \ldots, (\imath_N, p_N)$ of $(\Gamma_{\mathcal{Q}})_0$ such that

$$\exists \text{ a path } (\imath_k, p_k) \rightsquigarrow (\imath_l, p_l) \text{ in } \Gamma_{\mathcal{Q}} \implies k > l.$$

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Let R be the root system of Δ. To the vertex $(\imath_k, p_k)$, we assign

$$s_{\imath_1} s_{\imath_2} \cdots s_{\imath_{k-1}}(\alpha_{\imath_k}) \in \mathsf{R},$$

where $\alpha_\imath \in \mathsf{R}$ is the simple root associated with $\imath \in \Delta_0$ and $s_\imath$ is the corresponding simple reflection in the Weyl group W of R.

Assigning positive roots

A compatible reading of Γ_Q is an ordering $(\imath_1, p_1), \ldots, (\imath_N, p_N)$ of $(\Gamma_Q)_0$ such that

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The sequence $\underline{\boldsymbol{i}} = (\imath_1, \ldots, \imath_N)$ gives a reduced word for the longest element $w_0 \in \mathsf{W}$, hence the roots defined above are all the positive roots of R, without repetition. Moreover, the assignment independs on the choice of compatible reading of $\Gamma_{\mathcal{Q}}$.

Example: type B_3

If $\mathcal{Q}$ is the Q-datum of type B_3 from the previous example, then $\Gamma_{\mathcal{Q}}$ is given as follows:

$(\imath \setminus p)$	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
1							1 1 1 1 1						
2					1 1 1 1 0				0 1 1 1 1				
3		0 0 1 0 0		1 1 0 0 0		0 0 1 1 0		0 1 0 0 0		0 0 1 1 1			
4			1 1 1 0 0				0 1 1 1 0				0 0 0 1 1		
5	1 0 0 0 0				0 1 1 0 0				0 0 0 1 0				0 0 0 0 1

In each vertex, we indicate the corresponding positive root of A_5.

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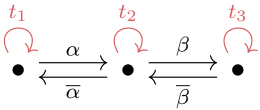

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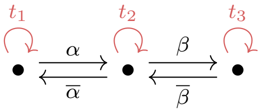

where black arrows have degree 0 and red arrows have degree -1. The differential is determined by $d(t_1 + t_2 + t_3) = [\alpha, \overline{\alpha}] + [\beta, \overline{\beta}]$.

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We can use $\mathrm{pvd}(\Pi)$ to categorify the previous constructions.

"Representations" of a Q-datum

Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum for $\mathfrak{g}$ and take a reduced word $\underline{\boldsymbol{i}} = (\imath_1, \ldots, \imath_N)$ for w_0 coming from a compatible reading of $\Gamma_{\mathcal{Q}}$.

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Proposition

If $\mathcal{Q} = Q$ is a Dynkin quiver of type ADE, then $\mathcal{C}(\mathcal{Q})$ is equivalent to $\operatorname{mod} KQ$ (as a K-linear category).

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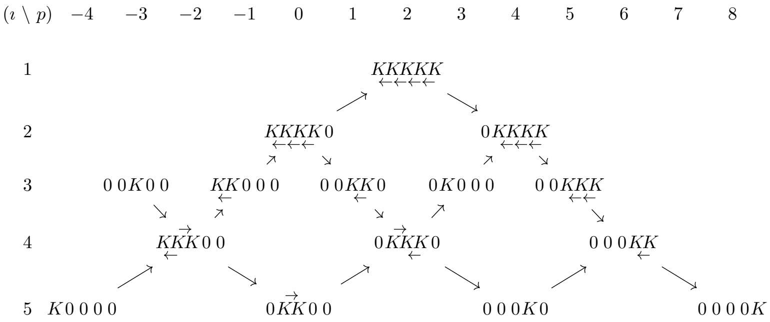
Theorem [C.]

The twisted AR quiver $\Gamma_{\mathcal{Q}}$ is isomorphic to the quiver obtained from the Gabriel quiver of $\mathcal{C}(\mathcal{Q})$ by removing all arrows parallel to paths of length ≥ 2.

Example: the category $\mathcal{C}(\mathcal{Q})$

If $\mathcal{Q}$ is the Q-datum of type B_3 from previous examples, then $\mathcal{C}(\mathcal{Q})$ can be described by the following picture.



Objects are depicted as preprojective representations. Arrows between objects correspond to irreducible morphisms in $\mathcal{C}(\mathcal{Q})$.

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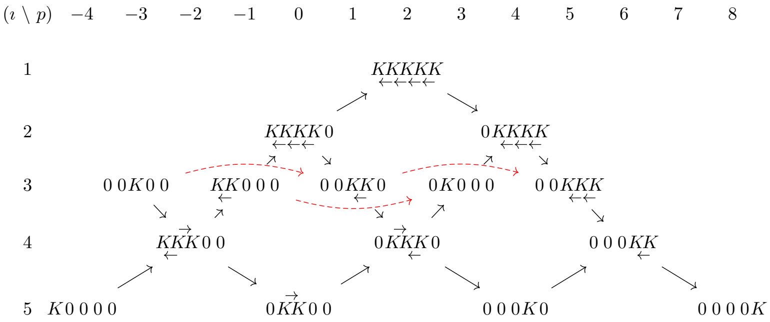

Objects are depicted as preprojective representations. Arrows between objects correspond to irreducible morphisms in $\mathcal{C}(\mathcal{Q})$.

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$$C = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad C(q) = \begin{bmatrix} q^2 + q^{-2} & -1 \\ -q - q^{-1} & q + q^{-1} \end{bmatrix}$$

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Example: $\widetilde{C}_{21}(q) = \frac{q^2+q^4}{1+q^6} = q^2 + q^4 - q^8 - q^{10} + q^{14} + \cdots$.

The simply-laced case

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One can naturally identify $\mathrm{ind}(\mathcal{D}^b(\mathrm{mod}\, KQ))$ with the set

$$\widehat{\Delta}_0 = \{(i,p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}.$$

Each $(i,p) \in \widehat{\Delta}_0$ gives an indecomposable object $H_Q(i,p)$.

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Theorem [Hernandez-Leclerc '15, Fujita '22]

For $(i,p),(j,s) \in \widehat{\Delta}_0$ with $s \geq p$, we have

$$\widetilde{c}_{ij}(s-p+1) = \langle H_Q(i,p), H_Q(j,s) \rangle$$

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The "derived category" of a Q-datum

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Proposition

If $\mathcal{Q} = Q$ is a Dynkin quiver of type ADE, then $\mathcal{D}(\mathcal{Q})$ is equivalent to $\mathcal{D}^b(\mathrm{mod}\, KQ)$ (as a K-linear category).

Final ingredients

- "Euler form": for $M, N \in \mathcal{D}(\mathcal{Q})$, we define

$$\langle M, N \rangle_{\mathcal{Q}} = \sum_{k \in \mathbb{Z}} (-1)^k \dim_K \operatorname{Ext}^k_{\mathcal{Q}}(M, N),$$

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Reinterpreting Fujita-Oh's formula

Theorem [Fujita-Oh '21, C.]

For $(i,p),(j,s) \in \widehat{I}$ with $p \geq s$ and $\max\{d_i, d_j\} = r$, we have

$$\widetilde{c}_{ij}(p-s+d_i) = \left\langle H_{\mathcal{Q}}(j,s), \bigoplus_{k=0}^{\lceil d_j/d_i \rceil -1} \tau_{\mathcal{Q}}^{k}(H_{\mathcal{Q}}(i,p)) \right\rangle_{\mathcal{Q}}.$$

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- Fujita-Oh's formula works without any restriction on d_i and d_j.
- In type B, the formula above also works in general.

Thank you for your attention!

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