Preprojective algebras, skew group algebras and Morita equivalences

(joint with X.-W. Chen, arXiv:2406.15049)

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Motivations

- Q: a finite acyclic quiver
- $\Pi(Q)$: the preprojective algebra of Q

$$\Pi(Q) = \mathbb{K}\overline{Q}/(\sum_{\alpha \in Q_1} \alpha\alpha^* - \alpha^*\alpha)$$

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I. M. GELFAND, AND V. A. PONOMAREV, Funct. Anal. Appl. **13** (1979), 157–166.

G. LUSZTIG, J. Amer. Math. Soc. **4** (2) (1991), 365–421.

C. M. RINGEL, In: Algebras and Modules II (Geiranger, 1996), CMS Conf. Proc. **24**, 467–480, Amer. Math. Soc. 1998.

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- [Buan-Iyama-Reiten-Scott, 2009;   Mizuno, 2014]

$$\Theta_Q : W(Q) \longrightarrow \langle I_i \mid i \in Q_0 \rangle$$

$\swarrow$ Weyl group of Q $\searrow$ Certain ideal monoid of $\Pi(Q)$

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$$\swarrow \qquad\qquad\qquad \searrow$$

Weyl group of Q   Certain ideal monoid of $\Pi(Q)$

- $I_i = \Pi(Q)(1 - e_i)\Pi(Q)$: the two-sided ideal of $\Pi(Q)$ generated by $1 - e_i, \forall i \in Q_0 \quad (\Pi(Q)/I_i \simeq S_i)$

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  - $C = (c_{ij}) \in M_n(\mathbb{Z})$: (symmetrizable) Cartan matrix
    - $c_{ii} = 2, \ \forall\, i$; $c_{ij} \leq 0, \ \forall\, i \neq j$, and $c_{ij} < 0 \Leftrightarrow c_{ji} < 0$;
    - $\exists\, D = \mathrm{diag}(c_1, \cdots, c_n)$ with $c_i \in \mathbb{Z}_{\geq 1}$ s.t. DC is symmetric.
  - $D = \mathrm{diag}(c_1, \cdots, c_n)$: a symmetrizer of C
  - Ω : an orientation of C : $\Omega \subset \{1, 2, \cdots, n\} \times \{1, 2, \cdots, n\}$ s.t.
    - $\{(i, j), (j, i)\} \cap \Omega \neq \emptyset \Leftrightarrow c_{ij} < 0$
    - $\forall \ (i_1, i_2, i_3, \cdots, i_{t+1})$ with $(i_s, i_{s+1}) \in \Omega$, we have $i_1 \neq i_{t+1}$.

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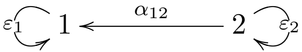
- Example $C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}$, $\Omega = \{(1,2)\}$

Motivations

- (C, D, Ω): a Cartan triple
- quiver of type C $\quad Q := Q(C, \Omega)$
  - $$Q_0 = \{1, 2, \ldots, n\};$$
  - $$Q_1 = \{\alpha_{ij}^{(g)} : j \to i \mid (i,j) \in \Omega, 1 \leq g \leq (c_{ij}, c_{ji})\} \cup \{\varepsilon_i : i \to i \mid 1 \leq i \leq n\}.$$
- Example

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}, \quad \Omega = \{(1,2)\}$$

$Q = Q(C, \Omega)$:



Motivations

- Geiss-Leclerc-Schröer (GLS) algebras :

(C, D, Ω) a Cartan triple, $Q = Q(C, \Omega)$.

$$H = H(C, D, \Omega) := kQ/I$$

I generated by

- $\varepsilon_i^{c_i} = 0, \ \forall\, i \in Q_0;$
- $\varepsilon_i^{\frac{c_i}{(c_i,c_j)}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{\frac{c_j}{(c_i,c_j)}}, \ \forall\, (i,j) \in \Omega, \ 1 \leq g \leq (c_{ij}, c_{ji}).$

C. Geiss, B. Leclerc, and J. Schröer, Invent. Math. **209** (2017), 61–158.

- Example

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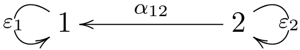
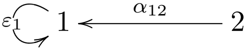


H : given by



$/ \ \varepsilon_1^2$

Motivations

Homological algebra aspects

- A : an $\mathbb{K}$-algebra, $A^e = A \otimes_{\mathbb{K}} A^{\mathrm{op}}$: the enveloping algebra of A
- The 2-preprojective algebra of A is:

$$\Pi_2(A) = T_A(\mathrm{Ext}^1_{A^e}(A, A^e))$$

- Rmk : (1) $A = \mathbb{K}Q$, then $\Pi_2(A) = \Pi(Q)$
  (2) A : f.d. $\mathrm{Ext}^n_{A^e}(A, A^e) \simeq \mathrm{Ext}^n_A(DA, A)$
  (3) derived preprojective algebras (Calabi-Yau completion)

$$\Pi_2^{der}(A) = T_A(\Sigma \mathbb{R}\mathrm{Hom}_{A^e}(A, A^e))$$

O. Iyama, and S. Oppermann, Adv. Math. **244** (2013), 23–68.

C. M. Ringel, In: Algebras and Modules II (Geiranger, 1996), CMS Conf. Proc. **24**, 467–480, Amer. Math. Soc. 1998.

B. Keller, J. Reine Angew. Math. **654** (2011), 125–180.

Motivations

- (C, D, Ω): a Cartan triple
- $\Pi(C, D, \Omega)$: the generalized preprojective algebra of (C, D, Ω)
- Fact : $\Pi(C, D, \Omega) \simeq \Pi_2(H)$

C. GEISS, B. LECLERC, AND J. SCHRÖER, Invent. Math. **209** (2017), 61–158.

Motivations

- (C, D, Ω): a Cartan triple Λ : the index set of C, D
- $\Pi = \Pi(C, D, \Omega)$: the generalized preprojective algebra of (C, D, Ω)

- [Fu-Geng, 2019]

$$\Theta_C : W(C) \longrightarrow \langle L_{\mathbf{j}} \mid \mathbf{j} \in \Lambda \rangle;$$

$\swarrow$ $\searrow$

Weyl group of C Certain ideal monoid of $\Pi(C, D, \Omega)$

C. Fu, and S. Geng, Algebr. Represent. Theor. **22** (2019), 1239–1260.

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Weyl group of C Certain ideal monoid of $\Pi(C, D, \Omega)$

- $L_{\mathbf{j}} = \Pi(1 - e_{\mathbf{j}})\Pi$: the two-sided ideal of $\Pi(C, D, \Omega)$ generated by $1 - e_{\mathbf{j}}, \forall \mathbf{j} \in \Lambda \quad (\Pi / L_{\mathbf{j}} \simeq E_{\mathbf{j}})$

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Motivations

- Recall the first bijection

$$\Theta_Q : W(Q) \longrightarrow \langle I_i \mid i \in Q_0 \rangle; \quad s_i \mapsto I_i$$

$\swarrow$ $\searrow$

Weyl group of Q — Certain ideal monoid of $\Pi(Q)$

- Question: How to compare these bijections Θ_Q and Θ_C ?

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- Recall the first bijection

$$\Theta_Q : W(Q) \longrightarrow \langle I_i \mid i \in Q_0 \rangle; \quad s_i \mapsto I_i$$

$\swarrow$ Weyl group of Q $\qquad$ $\searrow$ Certain ideal monoid of $\Pi(Q)$

- Question: How to compare these bijections Θ_Q and Θ_C ?
- Idea: Use folding !

The folding process

Cartan triple (C, D, Ω) associated to (Q, G) :

- (Q, G) : finite acyclic quiver with G-action
- Construct (C, D, Ω) from (Q, G):
  - index C and D by $Q_0/G = \{\mathbf{i} \mid i \in Q_0\}$
  - D:
    $$c_{\mathbf{i}} := \frac{|G|}{|\mathbf{i}|}$$
  - C:
    $$c_{\mathbf{ij}} := -\frac{N_{\mathbf{ij}}}{|\mathbf{j}|},$$
    $N_{\mathbf{ij}} =$ the number of arrows in Q between $\mathbf{i}$ and $\mathbf{j}$
  - Ω: $(\mathbf{i}, \mathbf{j}) \in \Omega \Leftrightarrow \exists$ arrow from j' to i' for some $j' \in \mathbf{j}$, $i' \in \mathbf{i}$.

The folding process

Example

$$Q: \quad 2 \xrightarrow{\alpha} 1 \xleftarrow{\alpha'} 2', \quad \sigma \text{ interchange } \alpha \text{ and } \alpha'$$

and $G = \langle \sigma \mid \sigma^2 = 1 \rangle$

- $Q_0/G = \{\mathbf{1}, \mathbf{2}\}$
- $c_{\mathbf{1}} = \frac{|G|}{|\mathbf{1}|} = 2, \; c_{\mathbf{2}} = \frac{|G|}{|\mathbf{2}|} = 1, \; D = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}$
- $c_{\mathbf{12}} = -\frac{2}{|\mathbf{2}|} = -1, \; c_{\mathbf{21}} = -\frac{2}{|\mathbf{1}|} = -2, \; C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$
- $\Omega = \{(\mathbf{1}, \mathbf{2})\}$.

The folding process

- Folding process

$$(Q, G) \rightsquigarrow (C, D, \Omega)$$

$\swarrow$ quiver with G-action

$\searrow$ Cartan triple

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$$(Q, G) \rightsquigarrow (C, D, \Omega)$$

$\swarrow$ quiver with G-action $\qquad$ $\searrow$ Cartan triple

- $\exists$ well-known isomorphism

$$\psi : W(C) \longrightarrow W(Q)^G; \quad r_{\mathbf{j}} \mapsto \prod_{i \in \mathbf{j}} s_i; \quad \forall\, \mathbf{j} \in Q_0/G.$$

R. Steinberg, Lectures on Chevalley Groups, Yale University, 1967.

J. Y. Hee, Geom. Dedicata **37** (1991), 65–102.

Motivations

(• Question: How to compare these bijections Θ_Q and Θ_C ?)

- Question : (more precisely)

$$\begin{array}{ccc}
W(C) & \xrightarrow{\quad\psi\quad} & W(Q)^G \\
\Theta_C \Big\downarrow & & \Big\downarrow \Theta_Q^G \\
\langle L_{\mathbf{j}} \mid \mathbf{j} \in Q_0/G \rangle & \overset{?\ \exists\ \Psi}{\dashrightarrow} & \langle I_i \mid i \in Q_0 \rangle^G
\end{array}$$

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- Answer : $\exists\ \Psi$ monoid isomorphism

Main result 1

(Q, G) : finite acyclic with G-action

(C, D, Ω) : Cartan triple associated to (Q, G)

Proposition

$\exists!$ iso. Ψ of monoids s.t. the diagram commute:

$$\begin{array}{ccc} W(C) & \xrightarrow{\psi} & W(Q)^G \\ \Big\downarrow{\scriptstyle \Theta_C} & & \Big\downarrow{\scriptstyle \Theta_Q^G} \\ \langle L_{\mathbf{j}} \mid \mathbf{j} \in Q_0/G \rangle & \xrightarrow{\Psi} & \langle I_i \mid i \in Q_0 \rangle^G \end{array}$$

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$$(Q, G) \overset{\text{folding process}}{\rightsquigarrow} (C, D, \Omega)$$

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$$\begin{array}{ccc} (Q, G) & \overset{\text{folding process}}{\rightsquigarrow} & (C, D, \Omega) \\ & & \big\downarrow {\scriptstyle \text{GLS},17} \\ & & H = H(C, D, \Omega) \end{array}$$

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(Q, G) & \xrightarrow{\text{folding process}} & (C, D, \Omega) \\
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X. W. Chen, and R. Wang, Proc. Royal Soc. Edinb. Math., 1–45, DOI: https://doi.org/10.1017/prm.2024.34.

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\Big\downarrow & & \Big\downarrow \text{GLS},17 \\
\Pi_2(\mathbb{K}Q\#G) & & \Pi_2(H) = \Pi(C, D, \Omega)
\end{array}
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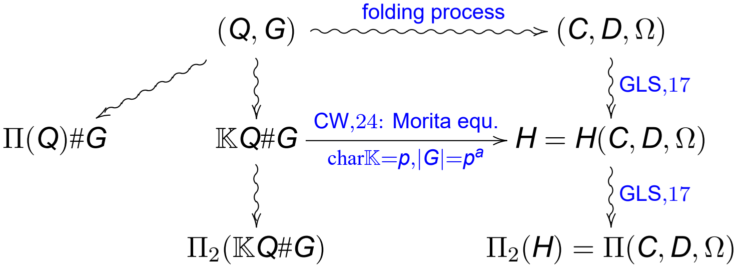
Motivations



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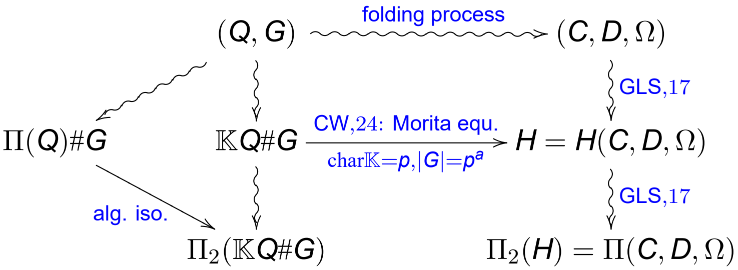

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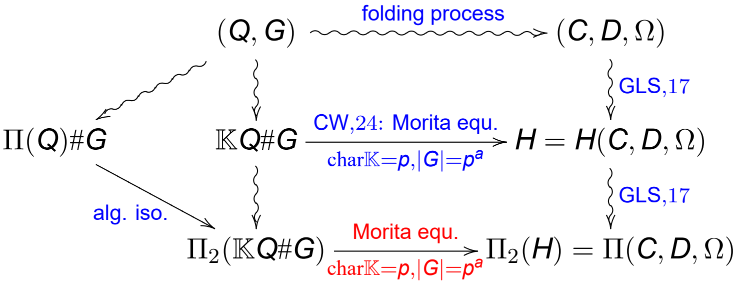

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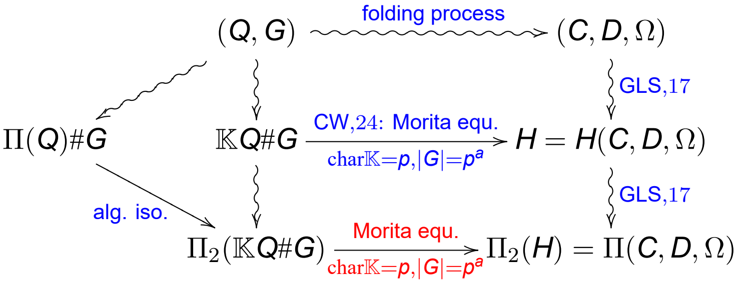
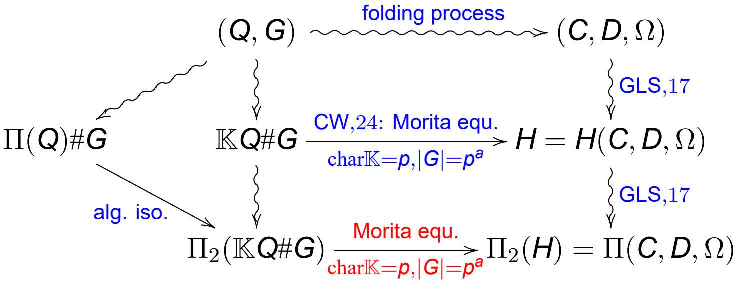


$\Rightarrow\ \exists$ Morita equ. : $\Pi(Q)\#G \xrightarrow{\sim} \Pi(C, D, \Omega)$

Motivations



$\Rightarrow\ \exists$ Morita equ. : $\Pi(Q)\#G \xrightarrow{\sim} \Pi(C, D, \Omega)$

$\Rightarrow\ \exists\ F : \Pi(Q)\#G\text{-Mod} \xrightarrow{\sim} \Pi(C, D, \Omega)\text{-Mod}$ equ. functor

Motivations

- $$\begin{array}{ccc} W(C) & \xrightarrow{\psi} & W(Q)^G \\ \Big\downarrow \Theta_C & & \Big\downarrow \Theta_Q^G \\ \langle L_{\mathbf{j}} \mid \mathbf{j} \in Q_0/G \rangle & \xrightarrow{\Psi} & \langle I_i \mid i \in Q_0 \rangle^G \end{array}$$

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- $F: \Pi(Q)\#G\text{-Mod} \xrightarrow{\sim} \Pi(C, D, \Omega)\text{-Mod}$ induces

$$F: \langle I_i \mid i \in Q_0 \rangle^G \# G \xrightarrow{\sim} \langle L_{\mathbf{j}} \mid \mathbf{j} \in Q_0/G \rangle$$
$$I\#G \mapsto \Psi^{-1}(I); \quad \forall\, I \in \langle I_i \mid i \in Q_0 \rangle^G$$

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- Ψ^{-1} is "categorified" by Morita equ F and induction $-\#G$

Main result 2

(C, D, Ω) : Cartan triple associated to (Q, G)

$(*)$: $G_\alpha = G_{s(\alpha)} \cap G_{t(\alpha)}$; $\forall \alpha \in Q$

Theorem

$\mathrm{char}(\mathbb{K}) = p > 0$, *$G$: cyclic p-group satisfying* $(*)$.
Then $\exists$ Morita equivalence

$$F\colon \Pi(Q)\#G\text{-Mod} \longrightarrow \Pi(C, D, \Omega)\text{-Mod}$$

s.t. the diagram commutes:

$$\begin{array}{ccc} W(C) & \xrightarrow{\quad \psi \quad} & W(Q)^G \\ \Big\downarrow {\scriptstyle \Theta_C} & & \Big\downarrow {\scriptstyle \Theta_Q^G} \\ \langle L_{\mathbf{i}} \mid \mathbf{i} \in Q_0/G \rangle & \xleftarrow{\; F \circ (-\#G) \;} & \langle I_i \mid i \in Q_0 \rangle^G \end{array}$$

Main results 2

Rmk : We mention that such a Morita equivalence F is established in [KKKMM24] independently by a different method, when Q is of type A and G is of order 2.

Y. Kimura, R. Koshio, Y. Kozakai, H. Minamoto, and Y. Mizuno, *τ-tilting theory and silting theory of skew group algebra extensions*, arXiv:2407.06711, 2024.

Example

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and $G = \langle \sigma \mid \sigma^2 = 1 \rangle$

- $\Pi(Q)$:

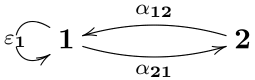
$$2 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 1 \underset{\alpha'^*}{\overset{\alpha'}{\leftrightarrows}} 2' \quad / \ \alpha^*\alpha,\ \alpha'^*\alpha',\ \alpha\alpha^* + \alpha'\alpha'^*.$$

- Its associated Cartan triple

$$(C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, D = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}, \Omega = \{(\mathbf{1}, \mathbf{2})\})$$

Example

- $\Pi(C, D, \Omega)$: given by



with $\varepsilon_1^2 = 0$, $\varepsilon_1\alpha_{12}\alpha_{21} + \alpha_{12}\alpha_{21}\varepsilon_1 = 0$, and $\alpha_{21}\alpha_{12} = 0$.

- $\mathrm{char}(\mathbb{K}) = 2$,

$$F : \Pi(Q)\#G\text{-Mod} \overset{\text{Morita equ.}}{\longrightarrow} \Pi(C, D, \Omega)\text{-Mod}$$

$$F(I_1\#G) = L_1, \quad F((I_2 I_{2'})\#G) = L_2.$$

- Consider

$$\psi : W(C) \to W(Q)^G$$

$$\psi(r_1) = s_1, \quad \psi(r_2) = s_2 s_{2'} = s_{2'} s_2.$$

Thank you for your attention !