Equivariant approach to simple singularities

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1. Introduction

Motivation

Proposition (Reiten-Riedtmann)

Let A be an Artin algebra and G a finite group which acts on A. Then the equivariant category of A-modules category is equivalent to the category of the skew group algebra $A[G]$-modules.

$$(\text{Mod-}A)^G \simeq \text{Mod-}(A[G])$$

[Reiten-Riedtmann] I. Reiten and C. Riedtmann. Skew group algebras in the representation theory of Artin algebras. *J. Algebra*, 92(1):224–282,1985.

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Our original motivation is to investigate the equivariant category of the maximal Cohen-Macaulay modules category over a local algebra of simple singularity.

[Reiten-Riedtmann] I. Reiten and C. Riedtmann. Skew group algebras in the representation theory of Artin algebras. *J. Algebra*, 92(1):224–282,1985.

2. Equivariantization

Group action on a category

- G: a finite group
- $\mathcal{C}$: an arbitrary category
- a strict G-action on $\mathcal{C}$: there exists a group homomorphism

$$F: G \to \operatorname{Aut} \mathcal{C}, \quad g \mapsto F_g$$

  that is, $F_g \circ F_h = F_{gh}$,
- Obviously, F_e is isomorphic to the identity functor $\mathrm{id}_{\mathcal{C}}$

Equivariant category $\mathcal{C}^G$

- G-equivariant object (X, α): $X \in \mathcal{C}$ and $\alpha = (\alpha_g)_{g \in G}$ where $\alpha_g : X \to F_g(X)$ is an isomorphism satisfying

$$\alpha_{gh} = F_g(\alpha_h) \circ \alpha_g,$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\alpha_g} & F_g(X) \\ {\scriptstyle \alpha_{gh}}\downarrow & & \downarrow{\scriptstyle F_g(\alpha_h)} \\ F_{gh}(X) & = & F_{gh}(X) \end{array}$$

Equivariant category $\mathcal{C}^G$

- Morphism $f : (X, \alpha) \to (Y, \beta)$: $f : X \to Y$ in $\mathcal{C}$ satisfying

$$\beta_g \circ f = F_g(f) \circ \alpha_g,$$

i.e. the following diagram commutes:

$$\begin{CD} X @>{\alpha_g}>> F_g(X) \\ @V{f}VV @VV{F_g(f)}V \\ Y @>{\beta_g}>> F_g(Y) \end{CD}$$

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- Equivariant category $\mathcal{C}^G$: the category of G-equivariant objects

Dual action

- **k**: an algebraically closed field with characteristics 0
- $\widehat{G}$: the character group $\mathrm{Hom}(G, k^*)$
- $\mathcal{C}$: an additive **k**-category
- Set $F_\chi(X, \alpha) = (X, \chi \otimes \alpha)$, where $(X, \alpha) \in \mathcal{C}^G$ and the isomorphism $(\chi \otimes \alpha)_g : X \to F_g(X)$ equals to $\chi(g^{-1})\alpha_g$ for each $g \in G$

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- This yields an automorphism

$$F_\chi : \mathcal{C}^G \to \mathcal{C}^G$$

  which acts on morphisms by the identity
- $\{F_\chi | \chi \in \widehat{G}\}$ forms a strict $\widehat{G}$-action on $\mathcal{C}^G$

Dual theorem

Theorem (Chen-Chen-Ruan)

Let G be a finite abelian group which splits over $\mathbf{k}$. Assume that $\mathcal{C}$ is idempotent complete and there is a strict G-action on $\mathcal{C}$. Then there exists an equivalence of categories

$$\mathcal{C} \to (\mathcal{C}^G)^{\widehat{G}}$$

[Chen-Chen-Ruan] J. Chen, X.-W. Chen and S. Ruan. The dual actions, equivariant autoequivalences and stable tilting objects. *Ann. Inst. Fourier(Grenoble)*, 70(6):2677–2736, 2020.

Induction functor

- Induction functor Ind: $\mathcal{C} \to \mathcal{C}^G$
  1. on objects: $\mathrm{Ind}(X) = (\oplus_{h \in G} F_h(X), \varepsilon)$
     where the isomorphism
     $$\varepsilon_g : \oplus_{h \in G} F_h(X) \to F_g(\oplus_{l \in G} F_l(X))$$
     is a $|G| \times |G|$ matrix $(f_{l,h})_{l,h \in G}$ that $f_{l,h} = \delta_{l,g^{-1}h}\mathrm{id}$ for each $g \in G$
  2. on morphisms: $\mathrm{Ind}(\alpha) = \oplus_{h \in G} F_h(\alpha)$

Induction functor

Theorem (Reiten-Riedtmann)

Let $\mathcal{C}$ be a Hom-finite abelian $\mathbf{k}$-category and G a finite abelian group whose order is invertible in $\mathcal{C}$. Assume that there is a strict G-action on $\mathcal{C}$.

1. *If $0 \to X \to Y \to Z \to 0$ is an almost split sequence in $\mathcal{C}$, then $0 \to \mathrm{Ind}(X) \to \mathrm{Ind}(Y) \to \mathrm{Ind}(Z) \to 0$ is a direct sum of almost split sequences in $\mathcal{C}^G$.*
2. *If $X \to Y$ is a minimal left or right almost split map in $\mathcal{C}$, then $\mathrm{Ind}(X) \to \mathrm{Ind}(Y)$ is a direct sum of minimal left or right almost split maps in $\mathcal{C}^G$.*

[Reiten-Riedtmann] I. Reiten and C. Riedtmann. Skew group algebras in the representation theory of Artin algebras. *J. Algebra*, 92(1):224–282,1985.

Involution

- Assume that $H = \{e, \sigma\}$
- Involution F_σ: $F_\sigma \circ F_\sigma = \mathrm{id}_{\mathcal{C}}$
- $\{F_e = \mathrm{id}_{\mathcal{C}}, F_\sigma\}$ is a strict H-action on $\mathcal{C}$

Equivariant category by involution

- Denote by (X, α_σ) the equivariant object (X, α)

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Equivariant category by involution

- Denote by (X, α_σ) the equivariant object (X, α)

Proposition (Reiten-Riedtmann)

Let X be an indecomposable object in $\mathcal{C}$.

1. *If $X \ncong F_\sigma X$, then $\mathrm{Ind}(X)$ is indecomposable in $\mathcal{C}^H$ and*

$$\mathrm{Ind}(X) \cong \mathrm{Ind}(F_\sigma X);$$

2. *If $X \cong F_\sigma X$ with an isomorphism $\alpha_X : X \to F_\sigma X$ satisfying $F_\sigma(\alpha_X) \circ \alpha_X = \mathrm{id}_X$, then $\mathrm{Ind}(X)$ decomposes as*

$$\mathrm{Ind}(X) \cong (X, \alpha_X) \oplus (X, -\alpha_X).$$

[Reiten-Riedtmann] I. Reiten and C. Riedtmann. Skew group algebras in the representation theory of Artin algebras. *J. Algebra*, 92(1):224–282,1985.

3. Equivariant approach to simple singularities

Maximal Cohen-Macaulay modules

- P: a formal power series ring $\mathbf{k}\{z_0, z_1, \cdots, z_m\}$
- f: a non-zero element in the maximal ideal of P
- R: a local ring $P/(f)$
- We say an R-module M is maximal Cohen-Macaulay if

$$\mathrm{Ext}^i_R(\mathbf{k}, M) = 0 \quad \text{for all } i < m$$

Matrix factorization

- a matrix factorization of f: a pair of square matrices

$$(\varphi : \ P^a \to P^a, \psi : \ P^a \to P^a)$$

with entries in P satisfying $\varphi \circ \psi = \mathrm{id}_{P^a} = \psi \circ \varphi$

- a morphism between (φ, ψ) and (φ', ψ'): a pair of morphisms (α, β) which makes the following diagram commutes

$$\begin{CD}
P^a @>{\varphi}>> P^a @>{\psi}>> P^a \\
@V{\alpha}VV @VV{\beta}V @VV{\alpha}V \\
P^b @>{\varphi'}>> P^b @>{\psi'}>> P^b
\end{CD}$$

Important equivalences

- MCM(R): the category of maximal Cohen-Macaulay R-modules
- $\mathrm{MF}_P(f)$: the category of matrix factorizations of f

[Eisenbud] D. Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.*, 260(1):35–64,1980.

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Lemma (Eisenbud)

The cokernel functor induces equivalences

$$\mathrm{MF}_P(f)/\{(1,f)\} \simeq \mathrm{MCM}(R)$$

$$\mathrm{MF}_P(f)/\{(1,f),(f,1)\} \simeq \mathrm{MCM}(R)/\{R\}$$

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Simple singularities

- $c(f)$: the set of proper ideals I of P with $f \in I^2$
- A simple singularity: a local ring $R = P/(f)$ with $c(f)$ finite
- We say R is of finite CM-representation type if there are only a finite number of isomorphism classes of indecomposable maximal Cohen-Macaulay modules over R

Simple singularities

Proposition (Buchweitz-Greuel-Schreyer)

The following are equivalent:

1. R *is of finite CM-representation type*
2. R *is a simple singularity*
3. R *is isomorphic to* $\mathbf{k}\{z_0, z_1, \cdots, z_m\}/(f(z_0, z_1) + z_2^2 + \cdots + z_m^2)$ *where* f *is equal to one of the following polynomials:*

$$\begin{array}{lll} (A_n) & z_0^{n+1} + z_1^2 & n \geq 1 \\ (D_n) & z_0^{n-1} + z_0 z_1^2 & n \geq 4 \\ (E_6) & z_0^4 + z_1^3 & \\ (E_7) & z_0^3 z_1 + z_1^3 & \\ (E_8) & z_0^5 + z_1^3 & \end{array}$$

[Buchweitz-Greuel-Schreyer] R.-O. Buchweitz, G.-M. Greuel and F.-O. Schreyer. Cohen-Macaulay modules on hypersurface singularities. II. *Invent. Math.*, 88(1):165–182, 1987.

H-action on a simple singularity

- Let $R = P/(f)$ be a simple singularity
- $R_1 = P\{z\}/(f + z^2)$
- Consider the involution σ on R_1:

$$\sigma(x) = x \text{ for each } x \in P, \text{ and } \sigma(z) = -z$$

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$$\sigma(x) = x \text{ for each } x \in P, \text{ and } \sigma(z) = -z$$

- It induces a natural involution F_σ on $\mathrm{MCM}(R_1)$:

$$F_\sigma : \mathrm{MCM}(R_1) \to \mathrm{MCM}(R_1), \ M \mapsto {}^\sigma M,$$

where the twist module ${}^\sigma M = M$ as an abelian group and the new R_1-action $\circ$ is given by $m \circ r = m \cdot \sigma(r)$. Moreover, F_σ acts on morphisms by the identity.

Main result

Theorem

There is an equivalence of categories

$$\mathrm{MCM}(R_1)^H \simeq \mathrm{MF}_P(f)$$

Knörrer's equivalence

- $R_1[\sigma]$: the skew group algebra with respect to the σ-action on R_1
- $\mathrm{MCM}_\sigma(R_1)$: the category of $R_1[\sigma]$-modules which are maximal Cohen-Macaulay over R_1

[Knörrer] H. Knörrer. Cohen-Macaulay modules on hypersurface singularities. I. *Invent. Math.*, 88(1):153–164,1987.

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Proposition (Knörrer)

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[Knörrer] H. Knörrer. Cohen-Macaulay modules on hypersurface singularities. I. *Invent. Math.*, 88(1):153–164,1987.

A corollary

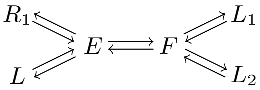
Corollary

There is an equivalence of categories

$$\mathrm{MCM}(R_1)^H \simeq \mathrm{MCM}_\sigma(R_1)$$

Example

- (Case D_5): $R = \mathbf{k}\{x, y\}/(x^2y+y^4)$, $R_1 = \mathbf{k}\{x, y, z\}/(x^2y+y^4+z^2)$
- the Auslander-Reiten quiver of $\mathrm{MCM}(R_1)$ is as following



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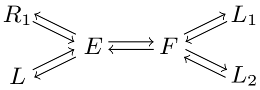
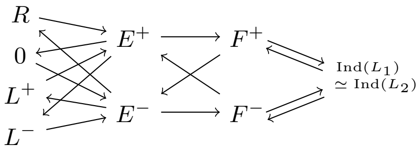


- F_σ-action: F_σ permutes L_1, L_2, and others are stable under F_σ
- the Auslander-Reiten quiver of $\mathrm{MF}_{\mathbf{k}\{x,y\}}(x^2y+y^4)$ is as following



Dual action on $\mathrm{MF}_P(f)$

- $\widehat{H}$: the character group $\{e, \chi\}$ of H
- Consider the involution $F_\chi : \mathrm{MF}_P(f) \to \mathrm{MF}_P(f)$ defined by

$$F_\chi(\varphi, \psi) = (\psi, \varphi), \quad F_\chi(\alpha, \beta) = (\beta, \alpha)$$

for any object (φ, ψ) and any morphism (α, β) in $\mathrm{MF}_P(f)$

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Proposition

There is an equivalence of categories

$$\mathrm{MF}_P(f)^{\widehat{H}} \simeq \mathrm{MCM}(R_1)$$

Thank you!