Representation type of cyclotomic quiver Hecke algebras[1]

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Outline

Background

Cyclotomic quiver Hecke algebra

a.k.a. cyclotomic Khovanov-Lauda-Rouquier algebra

- $U_q(\mathfrak{g})$: the quantum group of certain Kac-Moody algebra $\mathfrak{g}$
- $V(\Lambda)$: the integrable highest weight $U_q(\mathfrak{g})$-module with the highest weight Λ
- $\mathcal{R}^{\Lambda}$: the cyclotomic quiver Hecke algebra

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- $\mathcal{R}^\Lambda$: the cyclotomic quiver Hecke algebra

Lie Theory	Representation Theory
Weight spaces of $V(\Lambda)$	Blocks of $\mathcal{R}^\Lambda$
Crystal graph of $V(\Lambda)$	Socle branching rule for $\mathcal{R}^\Lambda$
Canonical basis in $V(\Lambda)$ over $\mathbb{C}$	Indecom. projective $\mathcal{R}^\Lambda$-modules
Action of the Weyl group of $\mathfrak{g}$ on $V(\Lambda)$	Derived equivalences between blocks of $\mathcal{R}^\Lambda$

Goal of Algebraic Representation Theory

Classify all indecomposable modules of a given algebra A and all morphisms between them, up to isomorphism.

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An algebra A is said to be

- **rep-finite** if the number of indecomposable modules is finite.
- **tame** if A is not rep-finite, but all indecomposable modules can be organized in a one-parameter family in each dimension.
- **wild** if there exists a faithful exact K-linear functor from the module category of $K\langle x, y\rangle$ to $\operatorname{mod} A$.

Representation type of algebra

Trichotomy Theorem (Drozd, 1977)

The representation type of an algebra A (over K) is exactly one of rep-finite, tame and wild.

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It leads to two directions:

(1) Studying mod A in-depth, such as Auslander-Reiten theory, homological dimensions, triangulated categories, etc, for rep-finite and tame algebras;

(2) Studying nice subcategories of mod A, such as Serre subcategories, wide subcategories, etc, for wild algebras.

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"The representation type of symmetric algebras is preserved under derived equivalence." *(Rickard 1991, Krause 1998)*

Cyclotomic quiver Hecke algebras

Lie theoretic data

Let $(\mathsf{A}, P, \Pi, P^\vee, \Pi^\vee)$ be the **Cartan datum** of type $X^{(1)}$, where

- $\mathsf{A} = (a_{ij})_{1 \le i.j \le \ell}$ is the Cartan matrix;
- $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_\ell \oplus \mathbb{Z}\delta$ is the weight lattice;
- $\Pi = \{\alpha_i \mid 0 \le i \le \ell\}$ is the set of simple roots;
- $P^\vee = \mathrm{Hom}(P, \mathbb{Z})$ is the coweight lattice;
- $\Pi^\vee = \{h_i \mid 0 \le i \le \ell\}$ is the set of simple coroots.

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We have

$$\langle h_i, \alpha_j \rangle = a_{ij}, \quad \langle h_i, \Lambda_j \rangle = \delta_{ij} \quad \text{for } 0 \le i, j \le \ell.$$

The null root is δ, e.g.,

$$\delta = \begin{cases} \alpha_0 + \alpha_1 + \cdots + \alpha_\ell & \text{if } X = A_\ell, \\ \alpha_0 + 2(\alpha_1 + \cdots + \alpha_{\ell-1}) + \alpha_\ell & \text{if } X = C_\ell. \end{cases}$$

Cyclotomic quiver Hecke algebra

The **cyclotomic quiver Hecke algebra** $R^{\Lambda}(\beta)$ with

$$\Lambda = a_0\Lambda_0 + \cdots + a_\ell\Lambda_\ell,\ \beta = b_0\alpha_0 + \cdots + b_\ell\alpha_\ell, \quad a_i, b_i \in \mathbb{Z}_{\geq 0},$$

is the K-algebra generated by

$$\{e(\nu) \mid \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in I^n\}, \quad \{x_i \mid 1 \leq i \leq n\}, \quad \{\psi_j \mid 1 \leq j \leq n-1\},$$

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subject to the following relations:

- $e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^n} e(\nu) = 1.$
- $x_1^{\langle h_{\nu_1},\Lambda\rangle}e(\nu) = 0, \quad x_ie(\nu) = e(\nu)x_i, \quad x_ix_j = x_jx_i.$
- $\psi_i^2e(\nu) = Q_{\nu_i,\nu_{i+1}}(x_i, x_{i+1})e(\nu), \quad \psi_ie(\nu) = e(s_i(\nu))\psi_i, \quad \psi_i\psi_j = \psi_j\psi_i$ if $|i-j| > 1$.
- $(\psi_ix_j - x_{s_i(j)}\psi_i)e(\nu) = \begin{cases} -e(\nu) & \text{if } j = i \text{ and } \nu_i = \nu_{i+1}, \\ e(\nu) & \text{if } j = i+1 \text{ and } \nu_i = \nu_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$
- $(\psi_{i+1}\psi_i\psi_{i+1} - \psi_i\psi_{i+1}\psi_i)e(\nu) = \begin{cases} \frac{Q_{\nu_i,\nu_{i+1}}(x_i,x_{i+1}) - Q_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})}{x_i - x_{i+2}}e(\nu) & \text{if } \nu_i = \nu_{i+2}, \\ 0 & \text{otherwise.} \end{cases}$

(1) $R^{\Lambda}(\beta)$ is a finite-dimensional symmetric algebra, see [Shan-Varagnolo-Vasserot, 2017].

(2) $R^{\Lambda}(\beta) \sim_{\text{derived}} R^{\Lambda}(\beta')$ if both $\Lambda - \beta$ and $\Lambda - \beta'$ lie in

$$\{\mu - m\delta \mid \mu \in \max^{+}(\Lambda), m \in \mathbb{Z}_{\geq 0}\},$$

see [Chuang-Rouquier, 2008].

(3) There is a bijection $\phi_{\Lambda} = \iota_{\Lambda} \circ \bar{\ } : \max^{+}(\Lambda) \to P_k^{+}(\Lambda)$, see [Kim-Oh-Oh, 2020].

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Set $\Lambda = m_{i_1}\Lambda_{i_1} + m_{i_2}\Lambda_{i_2} + \cdots + m_{i_n}\Lambda_{i_n}, m_{i_j} \neq 0$. Then,

$$|\Lambda| := m_{i_1} + \cdots + m_{i_j} \quad \text{and} \quad \text{ev}(\Lambda) := i_1 + \cdots + i_n.$$

In type $A_\ell^{(1)}$,

$$P_k^+(\Lambda) := \left\{\Lambda' \in P^+ \mid |\Lambda| = |\Lambda'| = k, \text{ev}(\Lambda) \equiv_{\ell+1} \text{ev}(\Lambda') \right\}.$$

Recall that $\langle h_i, \Lambda_j \rangle = \delta_{ij}$. We define $y_i := \langle h_i, \Lambda - \Lambda' \rangle$ and

$$Y_{\Lambda'} := (y_0, y_1, \ldots, y_\ell) \in \mathbb{Z}^{\ell+1}.$$

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Theorem (Ariki-Song-W., 2023)

The equation $AX^t = Y_{\Lambda'}^t$ has a unique solution $X = (x_0, x_1, \ldots, x_\ell)$ satisfying

$$x_i \geq 0 \quad \text{and} \quad \min\{x_i - \delta\} < 0.$$

Set $\beta_{\Lambda'} := x_0\alpha_0 + x_1\alpha_1 + \cdots + x_\ell\alpha_\ell$. Then,

$$\begin{aligned} \phi_\Lambda^{-1} : P_k^+(\Lambda) &\to \max{}^+(\Lambda) \\ \Lambda' &\mapsto \Lambda - \beta_{\Lambda'}. \end{aligned}$$

Constructions in affine type A

$$\Lambda' = \Lambda_i + \Lambda_j + \tilde{\Lambda} \in P_k^+(\Lambda) \Rightarrow \Lambda'_{i^-,j^+} := \Lambda_{i-1} + \Lambda_{j+1} + \tilde{\Lambda} \in P_k^+(\Lambda)$$

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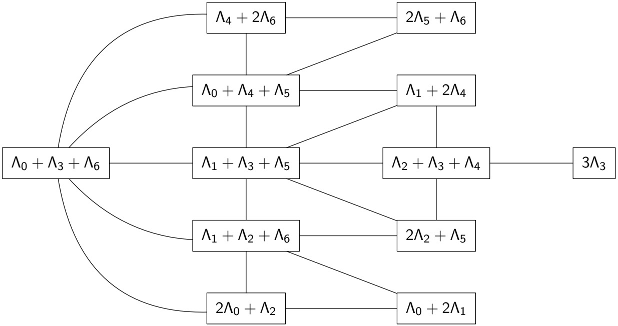
e.g., $P_3^+(\Lambda_0 + \Lambda_3 + \Lambda_6)$ in type $A_6^{(1)}$



We define

$$\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) & \text{if } i > j. \end{cases}$$

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We draw an arrow $\Lambda' \longrightarrow \Lambda'_{i^-,j^+}$ if

$$X_{\Lambda'} + \Delta_{i^-,j^+} = X_{\Lambda'_{i^-,j^+}}$$

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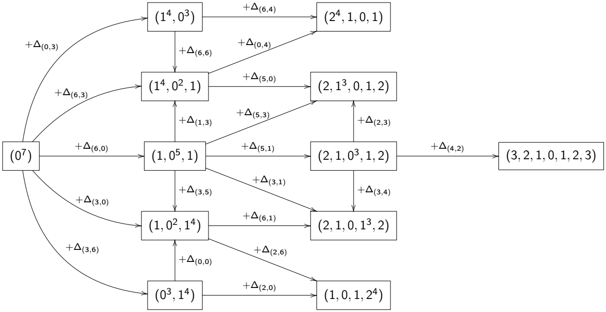

Key Lemmas

Lemma 1

The quiver $\vec{C}(\Lambda)$ of $P_k^+(\Lambda)$ is a finite connected quiver.

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Lemma 2

Suppose $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. There is a directed path

$$\Lambda^{(1)} \longrightarrow \Lambda^{(2)} \longrightarrow \cdots \longrightarrow \Lambda^{(m)} \in \vec{C}(\bar{\Lambda})$$

if and only if there is a directed path

$$\Lambda^{(1)} + \tilde{\Lambda} \longrightarrow \Lambda^{(2)} + \tilde{\Lambda} \longrightarrow \cdots \longrightarrow \Lambda^{(m)} + \tilde{\Lambda} \in \vec{C}(\Lambda).$$

Lemma 3

Write $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. If $R^{\bar{\Lambda}}(\beta)$ is representation-infinite (resp. wild), then $R^{\Lambda}(\beta)$ is representation-infinite (resp. wild).

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Lemma 4

Suppose that there is an arrow $\Lambda' \longrightarrow \Lambda''$ in $\vec{C}(\Lambda)$. If $R^{\Lambda}(\beta_{\Lambda'})$ is representation-infinite (resp. wild), then so is $R^{\Lambda}(\beta_{\Lambda''})$.

Rep-finite and tame sets in affine type A

Set $i_0 := i_h$, $i_{h+1} := i_1$ and write

$$\Lambda = m_{i_1}\Lambda_{i_1} + \cdots + m_{i_j}\Lambda_{i_j} + m_{i_{j+1}}\Lambda_{i_{j+1}} + \cdots + m_{i_h}\Lambda_{i_h}$$

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For any $1 \leq j \leq h$, we define

$$F(\Lambda)_0 := \left\{\Lambda_{i_j^-, i_j^+} \mid m_{i_j} = 2\right\}$$

$$F(\Lambda)_1 := \left\{\Lambda_{i_j^-, i_{j+1}^+} \mid m_{i_j} = 1, m_{i_{j+1}} = 1\right\}$$

$$T(\Lambda)_1 := \left\{\Lambda_{i_j^-, i_{j+1}^+} \mid m_{i_j} = 1, m_{i_{j+1}} > 1 \text{ or } m_{i_j} > 1, m_{i_{j+1}} = 1\right\}$$

$$T(\Lambda)_2 := \left\{(\Lambda_{i_j^-, i_j^+})_{(i_j-1)^-, (i_j+1)^+} \mid m_{i_j} = 2, i_{j-1} \not\equiv_{\ell+1} i_j - 1, i_{j+1} \not\equiv_{\ell+1} i_j + 1\right\} \text{ if } \operatorname{char} K \neq 2$$

$$T(\Lambda)_3 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_j^-, (i_j+1)^+ \text{ or } (i_j-1)^-, i_j^+} \mid m_{i_j} = 3, i_{j+1} \not\equiv_{\ell+1} i_j + 1 \text{ or } i_{j-1} \not\equiv_{\ell+1} i_j - 1\right\}$$
$$\text{if } \operatorname{char} K \neq 3$$

$$T(\Lambda)_4 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_j^-, i_j^+} \mid m_{i_j} = 4\right\} \text{ if } \operatorname{char} K \neq 2$$

$$T(\Lambda)_5 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_p^-, i_p^+} \mid m_{i_j} = m_{i_p} = 2, i_p \not\equiv_{\ell+1} i_j \pm 1, j \neq p\right\}$$

Set

$$\mathcal{F}(\Lambda) := \{\beta_{\Lambda'} \mid \Lambda' \in \{\Lambda\} \cup F(\Lambda)_0 \cup F(\Lambda)_1\},$$

$$\mathcal{T}(\Lambda) := \{\beta_{\Lambda'} \mid \Lambda' \in \cup_{1\leq j\leq 5} T(\Lambda)_j\}.$$

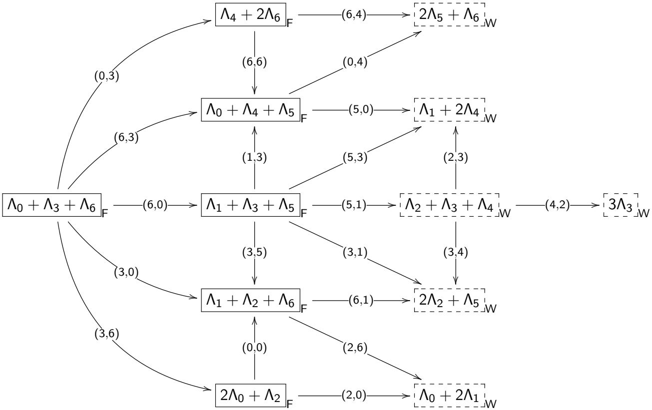
Theorem (Ariki-Song-W., 2023)

Suppose $|\Lambda| \geq 3$. Then, $R^\Lambda(\beta)$ is representation-finite if $\beta \in \mathcal{F}(\Lambda)$, tame if one of the following holds:

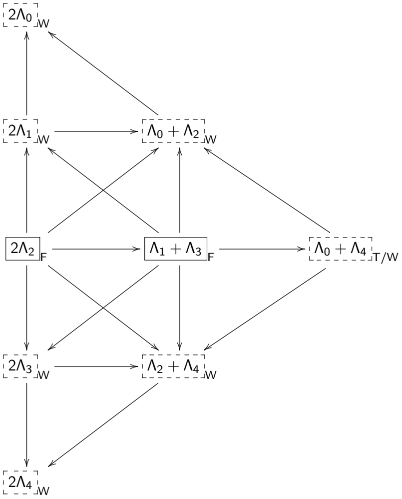
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell = 1$ with $t \neq \pm 2$,
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell \geq 2$ with $t \neq (-1)^{\ell+1}$,
- $\beta \in \mathcal{T}(\Lambda)$.

Otherwise, it is wild.

e.g., rep-type of $\vec{C}(\Lambda_0 + \Lambda_3 + \Lambda_6)$ in type $A_6^{(1)}$ is displayed as



e.g., rep-type of $\vec{C}(2\Lambda_2)$ in type $C_4^{(1)}$ is displayed as



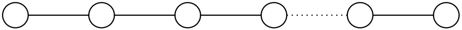
Derived Equivalence Class

Affine Type $\mathbb{A}$

Let $R^\Lambda(\beta)$ be the cyclotomic quiver Hecke algebra of type $A_\ell^{(1)}$.

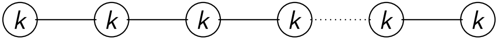
Theorem (Ariki-Song-W., 2023)

(1) If $R^\Lambda(\beta)$ is representation-finite, then it is derived equivalent to either $K[X]/(X^m)$ for $m \geq 1$ or a Brauer tree algebra whose Brauer tree is displayed as

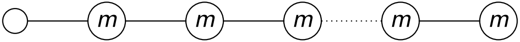


(2) If $R^\Lambda(\beta)$ is tame, then it is derived equivalent to one of

- $K[X,Y]/(X^3 - Y^3, XY)$, $K[X,Y]/(X^4 - Y^2, XY)$, $K[X,Y]/(X^2, Y^2)$, $K[X,Y]/(X^k - Y^k, XY)$ for $k \geq 3$.
- Brauer graph algebra associated with



  where $k = |\Lambda|$ and $\#\text{vertices} = \ell + 1$.
- Brauer graph algebra associated with



  where m and #vertices could be calculated explicitly.

Brauer graph algebra

Let A be a Brauer graph algebra with Brauer graph Γ_A.

Theorem (Antipov-Zvonareva, 2022)

If B is derived equivalent to A, then B is Morita equivalent to a Brauer graph algebra.

Theorem (Opper-Zvonareva, 2022)

$A \sim_{\text{derived}} B$ if and only if the following conditions hold.

(1) Γ_A and Γ_B share the same number of vertices, edges, faces,

(2) the multisets of multiplicities and the multisets of perimeters of faces of Γ_A and Γ_B coincide,

(3) either both or none of Γ_A and Γ_B are bipartite.

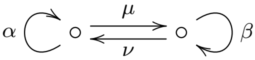
Affine Type $\mathbb{C}$

Let $R^\Lambda(\beta)$ be the cyclotomic quiver Hecke algebra of type $C_\ell^{(1)}$, where

$$\Lambda = \Lambda_0 + 2\Lambda_1, \quad \beta = \alpha_0 + \alpha_1.$$

Proposition (Ariki-Hudak-Song-W., 2024)

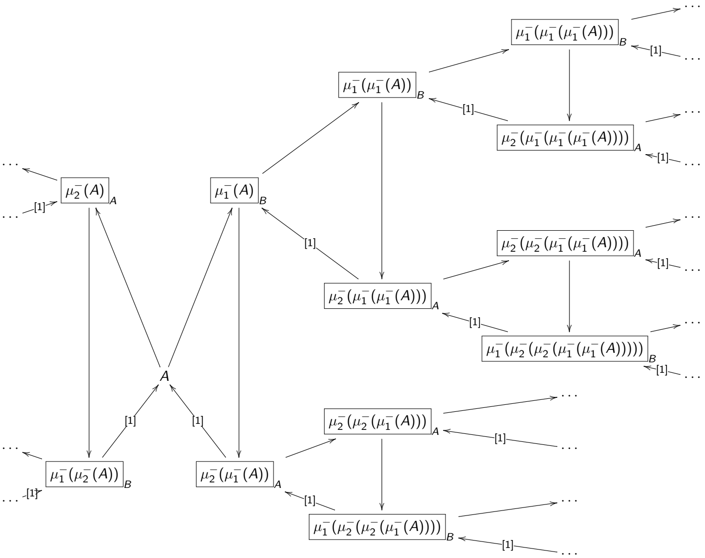
In this case, $R^\Lambda(\beta)$ is tame and it is Morita equivalent to the bound quiver algebra A with



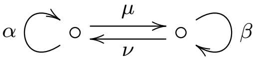
bounded by $\alpha^2 = 0, \beta^2 = \nu\mu, \alpha\mu = \mu\beta, \beta\nu = \nu\alpha$.

This is not a Brauer graph algebra!

Tilting quiver of A



Recall that

Q : 

,

and define

- $A := KQ/\langle \alpha^2, \beta^2 - \nu\mu, \alpha\mu - \mu\beta, \beta\nu - \nu\alpha \rangle$.
- $B := KQ/\langle \alpha^2 - \mu\nu, \beta^2 - \nu\mu, \alpha\mu - \mu\beta, \beta\nu - \nu\alpha, \mu\nu\mu, \nu\mu\nu \rangle$.

Proposition (Ariki-Hudak-Song-W., 2024)

If C is derived equivalent to A, then C is isomorphic to A or B.

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Thank you! Any questions?

Rule to draw arrows

Let Δ_{fin}^+ be the set of positive roots of the root system of type X.

- If $X = A_\ell$, $\Delta_{\mathrm{fin}}^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq \ell + 1\}$.
- If $X = B_\ell$, $\Delta_{\mathrm{fin}}^+ = \{\epsilon_i \mid 1 \leq i \leq \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.
- If $X = C_\ell$, $\Delta_{\mathrm{fin}}^+ = \{2\epsilon_i \mid 1 \leq i \leq \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.
- If $X = D_\ell$, $\Delta_{\mathrm{fin}}^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.

Rule to draw arrows

Let Δ^+_{fin} be the set of positive roots of the root system of type X.

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- If $X = B_\ell$, $\Delta^+_{\mathrm{fin}} = \{\epsilon_i \mid 1 \leq i \leq \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.
- If $X = C_\ell$, $\Delta^+_{\mathrm{fin}} = \{2\epsilon_i \mid 1 \leq i \leq \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.
- If $X = D_\ell$, $\Delta^+_{\mathrm{fin}} = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.

Then, the set $\Delta^+_{\mathrm{fin}} \sqcup (\delta - \Delta^+_{\mathrm{fin}})$ gives all arrows $\Lambda' \longrightarrow \Lambda''$.

Arrows in affine type A

Recall that $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_\ell = (1, 1, \ldots, 1)$. Then,

$$\Delta^+_{\text{fin}} \sqcup (\delta - \Delta^+_{\text{fin}}) = \{\epsilon_i - \epsilon_j, \delta - (\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq \ell + 1\}.$$

We have $\Delta_{i^-, j^+} =$

$$\begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) = \epsilon_i - \epsilon_{j+1} & \text{if } 0 < i \leq j \leq \ell, \\ (1^{j+1}, 0^{\ell-j}) = \delta - (\epsilon_{j+1} - \epsilon_{\ell+1}) & \text{if } 0 = i \leq j \leq \ell - 1, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) = \delta - (\epsilon_{j+1} - \epsilon_i) & \text{if } 0 \leq j < i \leq \ell. \end{cases}$$

Arrows in affine type C

Recall that $\delta = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell = (1, 2, \ldots, 2, 1)$.

- $\Delta_{i^+} = (1, 2^i, 1, 0^{\ell-i-1}) = \delta - (\epsilon_{i+1} + \epsilon_{i+2})$.

$$\Rightarrow \{\delta - (\epsilon_i + \epsilon_{i+1}) \mid 1 \leq i \leq \ell - 1\}.$$

- $\Delta_{i^-} = (0^{i-1}, 1, 2^{\ell-i}, 1) = \epsilon_{i-1} + \epsilon_i$.

$$\Rightarrow \{\epsilon_i + \epsilon_{i+1} \mid 1 \leq i \leq \ell - 1\}.$$

- $\Delta_{i^+, j^+} = (1, 2^i, 1^{j-i}, 0^{\ell-j})$ with $i + 1 \neq j$.

$$\Rightarrow \{\delta - (\epsilon_i + \epsilon_j) \mid 1 \leq i \leq j \leq \ell - 1, i + 1 \neq j\}.$$

- $\Delta_{i^-, j^-} = (0^i, 1^{j-i}, 2^{\ell-j}, 1)$ with $i + 1 \neq j$.

$$\Rightarrow \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j \leq \ell - 1, i + 1 \neq j\}.$$

- Δ_{i^-, j^+} with $i \neq 0, j \neq \ell, i - 1 \neq j$.

$$\Rightarrow \{\epsilon_i - \epsilon_j, \delta - (\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq \ell - 1\}.$$

We define

- $\Delta_{i^+} := (1, 2^i, 1, 0^{\ell-i-1}), \quad \Delta_{i^-} := (0^{i-1}, 1, 2^{\ell-i}, 1)$.
- $\Delta_{i^+,j^+} := (1, 2^i, 1^{j-i}, 0^{\ell-j}), \quad \Delta_{i^-,j^-} := (0^i, 1^{j-i}, 2^{\ell-j}, 1)$.
- $\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1, 2^j, 1^{i-j-1}, 2^{\ell-i}, 1) & \text{if } i \geq j+2. \end{cases}$

Set Δ and Λ'' for $\Lambda'_{i^\pm}$, $\Lambda'_{i^\pm,j^\pm}$, Λ'_{i^-,j^+}, respectively.

We define

- $\Delta_{i^+} := (1, 2^i, 1, 0^{\ell-i-1}), \quad \Delta_{i^-} := (0^{i-1}, 1, 2^{\ell-i}, 1).$
- $\Delta_{i^+,j^+} := (1, 2^i, 1^{j-i}, 0^{\ell-j}), \quad \Delta_{i^-,j^-} := (0^i, 1^{j-i}, 2^{\ell-j}, 1).$
- $\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1, 2^j, 1^{i-j-1}, 2^{\ell-i}, 1) & \text{if } i \geq j+2. \end{cases}$

Set Δ and Λ'' for $\Lambda'_{i^\pm}$, $\Lambda'_{i^\pm,j^\pm}$, Λ'_{i^-,j^+}, respectively.

We draw an arrow $\Lambda' \longrightarrow \Lambda''$ if

$$X_{\Lambda'} + \Delta = X_{\Lambda''}.$$

e.g., the quiver for $P_2^+(2\Lambda_2)$ in type $C_4^{(1)}$ is displayed as