F-invariant in cluster algebras

Peigen Cao

University of Science and Technology of China

August 6, 2024

The 21^{st} International Conference on Representations of Algebras
July 31-August 9, 2024, Shanghai

Plan

1. Cluster algebras: Definition and Laurent phenomenon
2. Two mutation invariants: tropical invariant and F-invariant
3. Oriented exchange graphs of cluster algebras

Plan

1. Cluster algebras: Definition and Laurent phenomenon
2. Two mutation invariants: tropical invariant and F-invariant
3. Oriented exchange graphs of cluster algebras

Compatible pair

- Fix $m, n \in \mathbb{Z}$ with $m \geq n > 0$
- $\widetilde{B} = \begin{bmatrix} B_{n\times n} \\ P \end{bmatrix}$: $m \times n$ integer matrix
- Λ: $m \times m$ integer **skew-symmetric matrix**

Definition (Compatible pair, Berenstein-Zelevinsky)

Call $(\widetilde{B}, \Lambda)$ a compatible pair, if there exists a **diagonal matrix** $S = diag(s_1, \ldots, s_n)$ with $s_i \in \mathbb{Z}_{>0}$ such that

$$\widetilde{B}^T \Lambda = (S \mid \mathbf{0})_{n\times m},$$

where $\mathbf{0}$ is the zero matrix of size $n \times (m-n)$.

Remark: The condition $\widetilde{B}^T \Lambda = (S \mid \mathbf{0})_{n\times m}$ implies $\widetilde{B}$ has the full rank n and $\widetilde{B}^T \Lambda \widetilde{B} = SB$ is skew-symmetric. Thus B is skew-symmetrizable.

Seed and Mutation

Call $t_0 = (X, \widetilde{B}, \Lambda)$ a seed in $\mathbb{F} := \mathbb{Q}(z_1, \ldots, z_m)$, if

- $X = (x_1, \ldots, x_m)$ is a free generating set of $\mathbb{F}$;
- $(\widetilde{B}, \Lambda)$ is a compatible pair.

In this case, call X a cluster and $x_1, \ldots, x_m$ cluster variables.

We have mutations $\mu_1, \ldots, \mu_n$ to produce new seeds:

$$\mu_k : (X, \widetilde{B}, \Lambda) \mapsto (X', \widetilde{B}', \Lambda'),$$

where $X' = (X \setminus \{x_k\}) \cup \{x'_k\}$ and the new cluster variable x'_k is determined by the k-th column of $\widetilde{B} = (b_{ij})$:

$$x'_k = x_k^{-1}\Big(\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}\Big) = \frac{\text{a binomial}}{x_k}.$$

Remarks on mutations

- μ_k is an involution, i.e.,

$$(X, \widetilde{B}, \Lambda) \overset{\mu_k}{\longmapsto} (X', \widetilde{B}', \Lambda') \overset{\mu_k}{\longmapsto} (X, \widetilde{B}, \Lambda)$$

- $(\widetilde{B}', \Lambda')$ is still a compatible pair for the same diagonal matrix, i.e.,

$$\widetilde{B}^T \Lambda = (S \mid \mathbf{0}) = (\widetilde{B}')^T \Lambda'.$$

- For the case of $m = n = 2$, $\mu_k : (\widetilde{B}, \Lambda) \mapsto (-\widetilde{B}, -\Lambda)$.

Example: Type A_2

Take $\widetilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \Lambda$ and $X = (x_1, x_2)$. Clearly, $m = n = 2$ and

$$\widetilde{B}^T \Lambda = (S \mid \mathbf{0}) = I_2.$$

$$(x_1, x_2) \xrightarrow{\mu_1} ({\color{red}x_3}, x_2) \xrightarrow{\mu_2} (x_3, {\color{red}x_4}) \xrightarrow{\mu_1} ({\color{red}x_5}, x_4) \xrightarrow{\mu_2} \dots$$

By applying the mutation relations:

$$x_{k+2} = \frac{x_{k+1} + 1}{x_k},$$

all cluster variables that we can obtain are as follows:

$${\color{red}x_3 = \frac{x_2 + 1}{x_1}}, \quad {\color{red}x_4 = \frac{x_1 + x_2 + 1}{x_1 x_2}}, \quad {\color{red}x_5 = \frac{x_1 + 1}{x_2}}$$

$$x_6 = x_1, \qquad x_7 = x_2 \quad \Longrightarrow \quad x_{i+5} = x_i, \quad \forall i.$$

Cluster algebra

Fix an initial seed $t_0 = (X, \widetilde{B}, \Lambda)$. We can get a collection of seeds:

$$\Delta = \{t = \overleftarrow{\mu}(t_0) \mid \overleftarrow{\mu} \text{ any sequence of mutations}\}.$$

Cluster algebra $\mathcal{A} = \mathcal{A}(t_0)$ is the $\mathbb{Z}$-subalgebra of $\mathbb{F} = \mathbb{Q}(x_1, \ldots, x_m)$ generated by all cluster variables in Δ.

Example (Type A_2, continued)

Take $\widetilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \Lambda$, $X = (x_1, x_2)$. Set $t_0 = (X, \widetilde{B}, \Lambda)$. Since all cluster variables in Δ are

$$x_1, \quad x_2, \quad x_3 = \frac{x_2 + 1}{x_1}, \quad x_4 = \frac{x_1 + x_2 + 1}{x_1 x_2}, \quad x_5 = \frac{x_1 + 1}{x_2},$$

$\mathcal{A} = \mathbb{Z}[x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1 x_2}, \frac{x_1+1}{x_2}] \subseteq \mathbb{Q}(x_1, x_2)$. Actually, $\mathcal{A} \subseteq \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$.

Laurent phenomenon and canonical expression

A cluster monomial u is a monomial in cluster variables from the **same** cluster.

Theorem (Fomin-Zelevinsky, Goss-Hacking-Keel-Kontsevich)

Let u be a cluster monomial and $t = (X_t, \widetilde{B}_t, \Lambda_t)$ a seed of $\mathcal{A}$. Then

(i) *The expansion of u w.r.t. X_t is a Laurent polynomial.*

(ii) *Set $\widehat{y}_{k;t} = X_t^{\widetilde{B}_t \mathbf{e}_k}$. The expansion above has a* **canonical expression**

$$u = X_t^{\mathbf{g}_u^t} F_u^t(\widehat{Y}_t) = X_t^{\mathbf{g}_u^t} F_u^t(\widehat{y}_{1;t}, \ldots, \widehat{y}_{n;t}),$$

where $\mathbf{g}_u^t \in \mathbb{Z}^m$ and $F_u^t \in \mathbb{Z}[y_1, \ldots, y_n]$ with $\mathbf{y_i} \nmid \mathbf{F_u^t}$, $\forall\ i$.

(iii) *F_u^t has positive coefficients and constant term 1.*

Call $\mathbf{g}_u^t \in \mathbb{Z}^m$ the extended g-vector, F_u^t the F-polynomial of u w.r.t seed t.

Example: canonical expression

Example (Type A_2, canonical expression)

Take $\widetilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \Lambda$, $X = (x_1, x_2)$. Set $t_0 = (X, \widetilde{B}, \Lambda)$. Then

$$\begin{aligned} x_1 &= x_1 \cdot 1, \\ x_2 &= x_2 \cdot 1, \\ x_3 &= \frac{x_2 + 1}{x_1} = x_1^{-1} x_2 \cdot (1 + \widehat{y}_1), \\ x_4 &= \frac{x_1 + x_2 + 1}{x_1 x_2} = x_1^{-1} \cdot (1 + \widehat{y}_1 + \widehat{y}_1 \widehat{y}_2), \\ x_5 &= \frac{x_1 + 1}{x_2} = x_2^{-1} \cdot (1 + \widehat{y}_2), \end{aligned}$$

where $\widehat{y}_1 = X^{\widetilde{B}\mathbf{e}_1} = x_2^{-1}$, $\quad \widehat{y}_2 = X^{\widetilde{B}\mathbf{e}_2} = x_1$.

Plan

1. Cluster algebras: Definition and Laurent phenomenon
2. Two mutation invariants: tropical invariant and F-invariant
3. Oriented exchange graphs of cluster algebras

Will **use the canonical expressions** to define two mutation invariants: $\langle -, - \rangle$ and $(- \mid\mid -)_F$, called tropical invariant and F-invariant.

Relationship: $(u \mid\mid u')_F = \langle u, u' \rangle + \langle u', u \rangle$ take symmetrized sum.

Tropical polynomial

Given a non-zero polynomial $F = \sum_{\mathbf{v}\in\mathbb{N}^n} c_{\mathbf{v}} Y^{\mathbf{v}} \in \mathbb{Z}[y_1, \ldots, y_n]$ and a vector $\mathbf{r} \in \mathbb{Z}^n$, denote by

$$F[\mathbf{r}] := \max\{\mathbf{v}^T\mathbf{r} \mid c_{\mathbf{v}} \neq 0\} \in \mathbb{Z}.$$

Call the map $F[-] : \mathbb{Z}^n \to \mathbb{Z}$ a tropical polynomial.

Key point: Replace a monomial "$Y^{\mathbf{v}}$" by a inner product "$\mathbf{v}^T\mathbf{r}$" and replace "$+$" by taking $\max\{-,-\}$.

Example

Take $F = 1 + y_1 + y_1y_2$ and $\mathbf{r} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Then

$$\begin{aligned} F[\mathbf{r}] &= \max\left\{[0,0]\begin{bmatrix} -1 \\ 1 \end{bmatrix},\ [1,0]\begin{bmatrix} -1 \\ 1 \end{bmatrix},\ [1,1]\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \\ &= \max\{0,\ -1,\ 0\} = 0. \end{aligned}$$

Remark: If F has constant term 1, then $F[\mathbf{r}] \geq 0,\ \forall\, \mathbf{r} \in \mathbb{Z}^n$.

Tropical invariant: $\langle -, - \rangle$

- Let $u = X_t^{\mathbf{g}_u^t} F_u^t(\widehat{Y}_t)$ and $u' = X_t^{\mathbf{g}_{u'}^t} F_{u'}^t(\widehat{Y}_t)$ be two **cluster monomials** of $\mathcal{A}$ written their **canonical expressions** in a seed $t = (X_t, \widetilde{B}_t, \Lambda_t)$.
- Using the canonical expressions, we define an integer:

$$\langle u, u' \rangle_t := (\mathbf{g}_u^t)^T \Lambda_t \mathbf{g}_{u'}^t + F_u^t[(S \mid \mathbf{0})\mathbf{g}_{u'}^t],$$

where the 2nd term is the tropical F-polynomial $F_u^t[-]$ valued at the modified g-vector $(S \mid \mathbf{0})\mathbf{g}_{u'}^t \in \mathbb{Z}^n$ of u' w.r.t. seed t.

Theorem (Tropical invariant, Cao)

The integer $\langle u, u' \rangle_t$ only depends on u and u', not on the choice of t.

P. Cao, *F*-invariant in cluster algebras, arXiv:2306.11438.

Tropical invariant: $\langle -, - \rangle$

Theorem (Tropical invariant, Cao)

The integer $\langle u, u' \rangle_t := (\mathbf{g}_u^t)^T \Lambda_t \mathbf{g}_{u'}^t + F_u^t[(S \mid \mathbf{0})\mathbf{g}_{u'}^t]$ *only depends on* u *and* u'*, not on the choice of* t*.*

Proof: Consider the g-vectors of u' w.r.t. different seeds:

$$\{\mathbf{g}_{u'}^w \in \mathbb{Z}^m \mid w \in \Delta\} \rightsquigarrow \{\Lambda_w \mathbf{g}_{u'}^w \in \mathbb{Z}^m \mid w \in \Delta\}.$$

Denote by $\mathbb{Q}_{sf}(x_1, \ldots, x_m) = \{P/Q \mid 0 \neq P, Q \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_m]\}$. Clearly, $\mathbb{Q}_{sf}(x_1, \ldots, x_m) = \mathbb{Q}_{sf}(X_w), \ \forall \ w \in \Delta$.

Claim: There exists a unique semifield homomorphism associated to u':

$$\beta_{u'} : (Q_{sf}(x_1, \ldots, x_m), \ \cdot \ , +) \to (\mathbb{Z}, \ +, \ \max\{-, -\})$$

s.t. $\beta_{u'}(X_w) = (\Lambda_w \mathbf{g}_{u'}^w)^T \in \mathbb{Z}^m_{\mathrm{row}}, \ \forall \ w \in \Delta$. Since $u = X_t^{\mathbf{g}_u^t} \bullet F_u^t(X_t^{\widetilde{B}_t})$, one has

$$\beta_{u'}(u) = (\mathbf{g}_u^t)^T \Lambda_t \mathbf{g}_{u'}^t \ + \ F_u^t[(S \mid \mathbf{0})\mathbf{g}_{u'}^t] = \langle u, u' \rangle_t$$

Notice that the left side only depends on u and u', not on the choice of t.

F-invariant: $(- \,||\, -)_F$

Definition (F-invariant, Cao)

The *F-invariant* between two cluster monomials u and u' is defined to be the **symmetrized sum**

$$
\begin{aligned}
(u \,||\, u')_F &= \langle u, u' \rangle_t + \langle u', u \rangle_t \\
&= (\mathbf{g}_u^t)^T \Lambda_t \mathbf{g}_{u'}^t + F_u^t[(S \mid \mathbf{0})\mathbf{g}_{u'}^t] + (\mathbf{g}_{u'}^t)^T \Lambda_t \mathbf{g}_u^t + F_{u'}^t[(S \mid \mathbf{0})\mathbf{g}_u^t].
\end{aligned}
$$

Since Λ_t is skew-symmetric, (1st term + 3rd term) $= 0$. Thus

$$(u \,||\, u')_F = F_u^t[(S \mid \mathbf{0})\mathbf{g}_{u'}^t] + F_{u'}^t[(S \mid \mathbf{0})\mathbf{g}_u^t].$$

(trop. F-polynomial of u valued at the modified g-vector of u' plus trop. F-polynomial of u' valued at the modified g-vector of u.)

Remark: Since F_u^t and $F_{u'}^t$ have constant term 1, $(u \,||\, u')_F \in \mathbb{Z}_{\geq 0}$.

F-invariant: Special case

Special case: If u and u' are two unfrozen cluster variables, say

$$u = x_{i;t},\ u' = x_{j;w},$$

where "unfrozen" means $i, j \in [1, n]$. Since $\mathbf{g}_u^t = \mathbf{e}_i \in \mathbb{Z}^m$ and $F_u^t = 1$, we have

$$\begin{aligned}(u \mid\mid u')_F &= F_u^t[(S \mid \mathbf{0})\mathbf{g}_{u'}^t] + F_{u'}^t[(S \mid \mathbf{0})\mathbf{g}_u^t] \\ &= 0 + F_{u'}^t[(S \mid \mathbf{0})\mathbf{e}_i] \\ &= \max\{\mathbf{v}^T(S \mid \mathbf{0})\mathbf{e}_i \mid c_\mathbf{v} \neq 0\} = s_i \cdot f_i',\end{aligned}$$

where s_i comes from $S = diag(s_1, \ldots, s_n)$ and f_i' is the maximal exponent of y_i in $F_{u'}^t = \sum_{\mathbf{v}\in\mathbb{N}^n} c_\mathbf{v} Y^\mathbf{v} \in \mathbb{Z}[y_1, \ldots, y_n]$. Thus

$$(u \mid\mid u')_F = (x_{i;t} \mid\mid x_{j;w})_F = s_i f_i' = s_i \cdot (x_{i;t} \mid\mid x_{j;w})_f,$$

where $(x_{i;t} \mid\mid x_{j;w})_f := f_i'$ is the f-compatibility degree defined by Fu-Gyoda using the components of **f**-vectors.

F-invariant: Special case

$$(x_{i;t} \mid\mid x_{j;w})_F = s_i \cdot (x_{i;t} \mid\mid x_{j;w})_f.$$

- f-compatibility degree $(- \mid\mid -)_f$ is defined on the set of unfrozen cluster variables;
- F-invariant $(- \mid\mid -)_F$ can be defined for any two cluster monomials and an important **advantage** of F-invariant is that we can calculate $(u \mid\mid u')_F$ by using any seed t.
- In fact, F-invariant can be defined for any two "good basis elements", e.g., theta functions constructed by Goss-Hacking-Keel-Kontsevich.

Theorem (Fu-Gyoda)

$(x_{i;t} \mid\mid x_{j;w})_f = 0$ iff $x_{i;t}$ and $x_{j;w}$ are contained in the same cluster.

Corollary: $(x_{i;t} \mid\mid x_{j;w})_F = 0$ iff $x_{i;t} \cdot x_{j;w}$ is a cluster monomial.

Recall that $(u \,||\, u')_F \in \mathbb{Z}_{\geq 0}$ for any two cluster monomials u and u'.

Now we discuss when $(u \,||\, u')_F = 0$.

Theorem (Cao)

For two cluster monomials u and u', their product $u \cdot u'$ is still a cluster monomial iff $(u \,||\, u')_F = 0$.

Proof: "$\Longrightarrow$" Say $u \cdot u'$ is a cluster monomial in seed t. Then $F_u^t = 1 = F_{u'}^t$. Thus

$$(u \,||\, u)_F = F_u^t[\ldots] + F_{u'}^t[\ldots] = 0.$$

"$\Longleftarrow$" (Not immediate!) ... and use the corollary of Fu-Gyoda's Theorem ...

Example: Type A_2

Take $\widetilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \Lambda$ and $X = (x_1, x_2)$. Clearly, $\widetilde{B}^T \Lambda = (S \mid \mathbf{0}) = I_2$.

$$(x_1, x_2) \xrightarrow{\mu_1} (x_3, x_2) \xrightarrow{\mu_2} (x_3, x_4) \xrightarrow{\mu_1} (x_5, x_4) \xrightarrow{\mu_2} (x_5, x_1) \xrightarrow{\mu_1} (x_2, x_1)$$

$$x_3 = x_1^{-1} x_2 \cdot (1 + \widehat{y}_1), \quad x_4 = x_1^{-1} \cdot (1 + \widehat{y}_1 + \widehat{y}_1 \widehat{y}_2), \quad x_5 = x_2^{-1} \cdot (1 + \widehat{y}_2).$$

By using the canonical expressions, we can calculate the F-invariant, e.g.,

$$\begin{aligned}
(x_3 \mid\mid x_4)_F &= F_{x_3}[\mathbf{g}_{x_4}] + F_{x_4}[\mathbf{g}_{x_3}] = (1 + y_1) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + (1 + y_1 + y_1 y_2) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \max\{0, \ -1\} + \max\{0, \ -1, \ 0\} = 0. \\
(x_3 \mid\mid x_5)_F &= F_{x_3}[\mathbf{g}_{x_5}] + F_{x_5}[\mathbf{g}_{x_3}] = (1 + y_1) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + (1 + y_2) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \max\{0, \ 0\} + \max\{0, \ 1\} = 1.
\end{aligned}$$

By the theorem, we know that $x_3 x_4$ is a cluster monomial, while $x_3 x_5$ is not.

Plan

1. Cluster algebras: Definition and Laurent phenomenon
2. Two mutation invariants: tropical invariant and F-invariant
3. Oriented exchange graphs of cluster algebras

Theorem: Oriented exchange graphs of (skew-symmetrizable) cluster algebras are acyclic.

Exchange graph and its oriented version

The exchange graph of a cluster algebra $\mathcal{A}$ is a **graph** Γ defined as follows:

- the vertex set of Γ is the set of seeds (up to permutations) of $\mathcal{A}$;
- the edges of Γ correspond to seed mutations.

Now we fix an initial seed $t_0 = (X, \widetilde{B}, \Lambda)$ of $\mathcal{A}$. By the sign-coherence of c-vectors, each seed mutation $t' = \mu_k(t)$ is either a green mutation or a red mutation, depending that the k-th column of the C-matrix $C_t^{t_0}$ lies in $\mathbb{Z}^n_{\geq 0}$ or $\mathbb{Z}^n_{\leq 0}$.

Remark: If $t' = \mu_k(t)$ is red mutation, then $t = \mu_k(t')$ is green mutation.

So each edge $t \overset{k}{\text{———}} t'$ in the exchange graph Γ has an orientation defined using green mutation. Thus we obtain a **quiver** $\overrightarrow{\Gamma}$, called the oriented exchange graph of $\mathcal{A}$.

Theorem: Oriented exchange graphs of (skew-symmetrizable) cluster algebras are acyclic.

Remark: The above result is well-known to experts for **skew-symmetric** cluster algebras, because in this case,

- the oriented exchange graphs of cluster algebras is a connected component of some oriented exchange graphs from representation theory;
- there is a natural **partial order** on the set of "seeds" (e.g., two-term silting objects, or τ-tilting pairs) on the representation theory side and thus their oriented exchange graphs are acyclic.

Remark: The partial order on τ-tilting pairs is induced by the partial order on the torsion classes.

Key point to the proof of the theorem: give a suitable replacement of torsion class on cluster algebras side so that we can define a partial order on cluster algebras side.

Dominant set of a seed (replacement of torsion class in τ-tilting thory)

- **Fix an initial seed** $t_0 = (X, \widetilde{B}, \Lambda)$. For a cluster monomial u, denote by

$$F_u := F_u^{t_0}, \quad \mathbf{g}_u := \mathbf{g}_u^{t_0},$$

 the F-polynomial and the extended g-vector of u with respect to the initial seed t_0.

- For a seed $t = (X_t, \widetilde{B}_t, \Lambda_t)$, denote by $u_t = \prod_{i=1}^n x_{i;t}$ the (basic) cluster monomial in t **with full support** on the unfrozen part.
- For a cluster variable z, recall $(z \mid\mid u_t)_F = F_z[(S \mid \mathbf{0})\mathbf{g}_{u_t}] + F_{u_t}[(S \mid \mathbf{0})\mathbf{g}_z]$.

Definition (Dominant set, Cao)

The dominant set of a seed t is defined to be

$$dom(u_t) := \{z \in \mathcal{X}^\circ \mid F_z[(S \mid \mathbf{0})\mathbf{g}_{u_t}] = 0\},$$

where $\mathcal{X}^\circ$ is the set of unfrozen cluster variables of $\mathcal{A} = \mathcal{A}(t_0)$.

Dominant set of a seed (replacement of torsion class in τ-tilting thory)

Keep $u_t = \prod_{i=1}^{n} x_{i;t}$ and $dom(u_t) := \{z \in \mathcal{X}^\circ \mid F_z[(S \mid \mathbf{0})\mathbf{g}_{u_t}] = 0\}$.

- If two seeds t and t' are the same up to a permutation, then $u_t = u_{t'}$ and thus $dom(u_t) = dom(u_{t'})$.
- Since the F-polynomial of an initial cluster variable is 1, we have

$$\{\text{initial unfrozen cluster variables}\} \subseteq dom(u_t)$$

  for any seed t.
- By ignoring the initial cluster variables, the dominant set $dom(u_t) = \{z \in \mathcal{X}^\circ \mid F_z[(S \mid \mathbf{0})\mathbf{g}_{u_t}] = 0\}$ of the seed t is "like" the set

$$\{\text{indecomposable } \tau\text{-rigid modules in } Fac M\}$$

  for a τ-tilting pair (M, P) in $mod A$. (Will give more details later.)

Oriented exchange graphs are acyclic

Theorem (Cao)

If $t' = \mu_k(t)$ is a green mutation in $\mathcal{A}$, then $dom(u_t) \subsetneq dom(u_{t'})$. In particular,

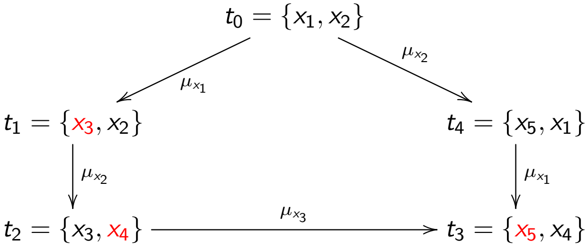
- *green mutations induce a partial order on the set of seeds (up to permutations) of $\mathcal{A}$;*
- *the oriented exchange graph of $\mathcal{A}$ is acyclic.*

Remark: The above result will appear in the second version of the paper:

P. Cao, *F*-invariant in cluster algebras, arXiv:2306.11438.

Example: Type A_2

Take $\widetilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \Lambda$ and $X = (x_1, x_2)$. Clearly, $\widetilde{B}^T \Lambda = (S \mid \mathbf{0}) = I_2$.



$$x_3 = x_1^{-1} x_2 \cdot (1 + \widehat{y}_1), \quad x_4 = x_1^{-1} \cdot (1 + \widehat{y}_1 + \widehat{y}_1 \widehat{y}_2), \quad x_5 = x_2^{-1} \cdot (1 + \widehat{y}_2).$$

One can check that

$$\begin{aligned} dom(u_{t_0}) &= \{x_1, x_2\}, \quad dom(u_{t_1}) = \{x_1, x_2, x_3\}, \quad dom(u_{t_2}) = \{x_1, x_2, x_3, x_4\} \\ dom(u_{t_3}) &= \{x_1, x_2, x_3, x_4, x_5\}, \quad dom(u_{t_4}) = \{x_1, x_2, x_5\}. \end{aligned}$$

End with some representation theory

The next few pages will answer the following question:

Why the dominant set

$$dom(u_t) = \{z \in \mathcal{X}^\circ \mid F_z[(S \mid \mathbf{0})\mathbf{g}_{u_t}] = 0\}$$

can be viewed as an replacement of torsion class in τ-tilting theory?

Remark: In our convention, the initial (unfrozen) cluster variables correspond to the τ-rigid pairs $(0, P_1), \ldots, (0, P_n)$.

Dominant set from the viewpoint of τ-tilting theory

Recall: "by ignoring the initial cluster variables, the dominant set $dom(u_t) = \{z \in \mathcal{X}^\circ \mid F_z[(S \mid \mathbf{0})\mathbf{g}_{u_t}] = 0\}$ of the seed t is "like" the set

$$\{\text{indecomposable } \tau\text{-rigid modules in } Fac M\}$$

for a τ-tilting pair (M, P) in $mod A$". Reason as follows:

- Let $\theta = \mathbf{g}_{(M,P)}$ be the g-vector of the τ-tilting pair (M, P), which is like the "negative" of the modified g-vector $(S \mid \mathbf{0})\mathbf{g}_{u_t}$ of $u_t = \prod_{i=1}^{n} x_{i;t}$.
- It is known from [Asai'2021], [Yurikusa'2018] that

  $Fac M = \{U \in mod A \mid \forall \text{ quotient module X of U}, \ \langle \theta, dim X \rangle \geq 0\} = \overline{\mathcal{T}}_\theta.$
- One notice: $\langle \theta, dim X \rangle = 0$ for the quotient module $X = 0$. So the condition $\langle -\theta, dim X \rangle = (dim X)^T(-\theta) \leq 0$ for each quotient module X of U is like

$$F_U[-\theta] = \max\{\mathbf{v}^T(-\theta) \mid c_\mathbf{v} \neq 0\} = 0,$$

  where "$F_U = \sum c_\mathbf{v} Y^\mathbf{v}$ " is the F-polynomial of U defined using quotient modules of U.

- In summary, $U \in Fac M \Longleftrightarrow F_U[-\theta] = 0$ and it is "like"

$$F_z[(S \mid \mathbf{0})\mathbf{g}_{u_t}] = 0$$

on the cluster algebras side.

- Hence, by ignoring the initial cluster variables, the dominant set

$$dom(u_t) = \{z \in \mathcal{X}^\circ \mid F_z[(S \mid \mathbf{0})\mathbf{g}_{u_t}] = 0\}$$

of the seed t is like the set

$$\{\text{indecomposable } \tau\text{-rigid modules in } Fac M\}$$

for a τ-tilting pair (M, P) in $mod A$, which is a kind of replacement of the torsion class $Fac M$.

Final Remark

F-invariant $(- \parallel -)_F$ is related to the following known invariants in cluster theory, because all these are related to the components of f-vectors.

(i) Fomin-Zelevinsky's compatibility degree $(- \parallel -)$ defined on almost positive roots of a Cartan matrix of finite type;

(ii) Fu-Gyoda's f-compatibility degree $(- \parallel -)_f$ defined on the set of unfrozen cluster variables, as mentioned before;

(iii) Derksen-Weyman-Zelevinsky's E-invariant $E^{\mathrm{sym}}(-,-)$ defined in the additive categorification of cluster algebras, which is closely related to the extension dimension in cluster categories;

(iv) Kang-Kashiwara-Kim-Oh's $\mathfrak{d}$-invariant $\mathfrak{d}(-,-)$ defined in the monoidal categorification of (quantum) cluster algebras, which is related to some information of the (renormalized) R-matrices $\mathbf{r}_{M,N}$ and $\mathbf{r}_{N,M}$ in the representation theory of quiver Hecke algebras.

Thank you!

S. Asai.
The wall-chamber structures of the real Grothendieck groups.
Adv. Math., 381:Paper No. 107615, 44 pp, 2021.

A. Berenstein and A. Zelevinsky.
Quantum cluster algebras.
Adv. Math., 195(2):405–455, 2005.

P. Cao.
F-invariant in cluster algebras.
arXiv: 2306.11438, 2023.

H. Derksen, J. Weyman, and A. Zelevinsky.
Quivers with potentials and their representations II: Applications to cluster algebras.
J. Amer. Math. Soc., 23(1):749–790, 2010.

C. Fu and Y. Gyoda.
Compatibility degree of cluster complexes.
Ann. Inst. Fourier, 74(2):663–718, 2024.

S. Fomin and A. Zelevinsky.
Cluster algebras I: Foundations.
J. Amer. Math. Soc., 15:497–529, 2002.

S. Fomin and A. Zelevinsky.
Y-systems and generalized associahedra.
Ann. Math., 158(3):977–1018, 2003.

S. Fomin and A. Zelevinsky.
Cluster algebras IV: Coefficients.
Compos. Math., 143(1):112–164, 2007.

M. Gross, P. Hacking, S. Keel, and M. Kontsevich.
Canonical bases for cluster algebras.
J. Amer. Math. Soc., 31(2):497–608, 2018.

S.-J. Kang, M. Kashiwara, M. Kim, and S.-J. Oh.
Monoidal categorification of cluster algebras.
J. Amer. Math. Soc., 31:349–426, 2018.

T. Yurikusa.
Wide subcategories are semistable.
Doc. Math., 23:35–47, 2018.

[Asa21], [BZ05], [Cao23], [DWZ10], [FZ02], [FZ03], [FZ07], [FG24],[GHKK18], [KKKO18], [Yur18]