Right triangulated quotient categories

Panyue Zhou

長沙理工大學
CHANGSHA UNIVERSITY OF SCIENCE & TECHNOLOGY

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Contents

Triangulated categories

Triangulated categories were introduced in the mid 1960's by J. L. Verdier in his thesis. Having their origins in algebraic geometry and algebraic topology, triangulated categories have by now become indispensable in many different areas of mathematics.

Nowadays there are important applications of triangulated categories in areas like algebraic geometry (derived categories of coherent sheaves, theory of motives) algebraic topology (stable homotopy theory), commutative algebra, differential geometry (Fukaya categories), microlocal analysis or representation theory (derived and stable module categories).

J. L. Verdier, Des catégories dérivées des catégories abéliennes. Astérisque, No. 239, 1996.

Part I: Right triangulated categories

Definition (Assem-Beligiannis-Marmaridis, 1994 and 1998)

Let $\mathcal{C}$ be an additive category and $\Sigma\colon \mathcal{C} \to \mathcal{C}$ be an additive endofunctor. (it is call suspension functor)

A sextuple (X, Y, Z, u, v, w) in $\mathcal{C}$ is a sequence of morphisms

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

with $X, Y, Z \in \mathcal{C}$. (it is called right triangle)

- I. Assem, A. Beligiannis, N. Marmaridis. Right triangulated categories with right semi-equivalences, in: Algebras and Modules, II, Geiranger, 1996, in: CMS Conf. Proc., vol. 24, Amer. Math. Soc., Providence, RI, 1998, pp. 17–37.
- A. Beligiannis, N. Marmaridis. Left triangulated categories arising from contravariantly finite subcategories. Comm. Algebra 22(12): 5021–5036, 1994.

Right triangulated categories

Definition (Assem-Beligiannis-Marmaridis, 1994 and 1998)

A morphism from sextuples (X, Y, Z, u, v, w) to (X', Y', Z', u', v', w') is a triple (α, β, γ) of morphisms of $\mathcal{C}$, which makes the following diagram commutative:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

If in this situation α, β and γ are isomorphisms in $\mathcal{C}$, then the morphism (α, β, γ) is called an isomorphism of sextuples.

Right triangulated categories

Definition (Assem-Beligiannis-Marmaridis, 1994 and 1998)

A right triangulated category is a triple $(\mathcal{C}, \Sigma, \nabla)$, where ∇ is a class of sextuples (whose elements are called right triangles), which satisfies the following axioms:

(RTR1) (a) The class ∇ is closed under isomorphisms.

(b) For each object $X \in \mathcal{C}$, the trivial sequence

$$0 \longrightarrow X \xrightarrow{1_X} X \longrightarrow 0$$

belongs to ∇.

(c) Every morphism $u\colon X \to Y$ in $\mathcal{C}$ can be embedded into a right triangle:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

Right triangulated categories

Definition (Assem-Beligiannis-Marmaridis, 1994 and 1998)

(RTR2) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a right triangle, then its rotation

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

is a right triangle.

(RTR3) Each solid commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

with rows in ∇ can be completed to a morphism of right triangles.

Right triangulated categories

Definition (Assem-Beligiannis-Marmaridis, 1994 and 1998)

(RTR4) (Octahedral axiom) Given right triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$,

$$B \xrightarrow{a} X \xrightarrow{b} Z \xrightarrow{c} \Sigma B \quad \text{and} \quad A \xrightarrow{af} X \xrightarrow{d} Y \xrightarrow{e} \Sigma A\ .$$

Then there are morphisms $s\colon C \to Y$ and $t\colon Y \to Z$ such that the following diagrams commute and the third column is a right triangle.

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\| & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle s} & & \| \\
A & \xrightarrow{af} & X & \xrightarrow{d} & Y & \xrightarrow{e} & \Sigma A \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \Sigma f} \\
 & & Z & = & Z & \xrightarrow{c} & \Sigma B \\
 & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle w:=\Sigma g\circ c} & & \\
 & & \Sigma B & \xrightarrow{\Sigma g} & \Sigma C & &
\end{array}$$

Some examples of right triangulated categories

Example

(1) Triangulated categories are right triangulated categories. Moreover, A right triangulated category $(\mathcal{C}, \Sigma, \nabla)$ is a triangulated category if and only if $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is an automorphism.

(2) Any additive category with cokernels (in particular any abelian category) is right triangulated category, where $\Sigma = 0$, and ∇ is the class of right exact sequences.

right exact sequence: $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is exact.

Some examples of right triangulated categories

Example (They come from some references)

(3) Let $\mathcal{C}$ be an abelian category and $\mathscr{X}$ be a covariantly finite subcategory of $\mathcal{C}$. Then the quotient category $\mathcal{C}/[\mathscr{X}]$ is a right triangulated category.

(4) Let $\mathscr{B}$ be an exact category with enough injectives. The subcategory of injective objects is denoted by $\mathcal{I}$. Then the quotient category $\overline{\mathscr{B}} = \mathscr{B}/[\mathcal{I}]$ is a right triangulated category.

(5) Let $\mathcal{C}$ be a triangulated category and $\mathscr{X}$ be a covariantly finite subcategory of $\mathcal{C}$. Then the quotient category $\mathcal{C}/[\mathscr{X}]$ is a right triangulated category.

Some examples of right triangulated categories

Example

(6) Let $(\mathcal{C}, \Sigma, \nabla)$ be a triangulated category and $\mathscr{X}$ be an extension closed subcategory of $\mathcal{C}$. If $\Sigma\mathscr{X} \subseteq \mathscr{X}$, then $\mathscr{X}$ is a right triangulated category.

(7) Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure in a triangulated category. Then $\mathcal{D}^{\leq 0}$ is a right triangulated category.

(8) Let $(\mathscr{X}, \mathscr{Y})$ be a co-t-structure in a triangulated category. Then $\mathscr{Y}$ is a right triangulated category.

Some properties of right triangulated categories

Proposition (Beligiannis-Marmaridis, 1998)

Suppose we are given a morphism of right triangles as in the following diagram.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A' \end{array}$$

If α and β are isomorphisms, then γ is also an isomorphism.

Proposition

Let $(\mathcal{C}, \Sigma, \nabla)$ be a right triangulated category. Then ∇ is closed under direct sums and direct summands.

In other words, the direct sum of two right triangles is still a right triangle, and the direct summand of a right triangle is also a right triangle.

Part III: Our main results

Definition

Let $\mathcal{C}$ be an additive category and $\Sigma\colon \mathcal{C} \to \mathcal{C}$ be an additive endofunctor.

We denote by ∇ a class of sextuples.

A triple $(\mathcal{C}, \Sigma, \nabla)$ is called pre-right triangulated category if ∇ satisfies axioms $(\mathrm{RTR1})$, $(\mathrm{RTR2})$ and $(\mathrm{RTR3})$.

Our main results

(RTR4-1) (Mapping cone axiom) Let $(\mathcal{C}, \Sigma, \nabla)$ be a pre-right triangulated category.

Given the solid part of the commutative diagram with rows in ∇.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A' \end{array}$$

Then there exists a morphism $\gamma\colon C \to C'$ such that each square commutes, and the mapping cone

$$B \oplus A' \xrightarrow{\begin{pmatrix} -g & 0 \\ \beta & f' \end{pmatrix}} C \oplus B' \xrightarrow{\begin{pmatrix} -h & 0 \\ \gamma & g' \end{pmatrix}} \Sigma A \oplus C' \xrightarrow{\begin{pmatrix} -\Sigma f & 0 \\ \Sigma\alpha & h' \end{pmatrix}} \Sigma B \oplus \Sigma A'$$

belongs to ∇.

Our main results

(RTR4-2) Let $(\mathcal{C}, \Sigma, \nabla)$ be a pre-right triangulated category.

Given the solid part of the commutative diagram with rows in ∇.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \| \\ A & \xrightarrow{a} & X & \xrightarrow{b} & Y & \xrightarrow{c} & \Sigma A \end{array}$$

Then there exists a morphism $\gamma\colon C \to Y$ such that each square commutes, and the sextuple

$$B \xrightarrow{\binom{g}{\beta}} C \oplus X \xrightarrow{(\gamma,\ -b)} Y \xrightarrow{\Sigma f \circ c} \Sigma B$$

belongs to ∇.

Our main results

Theorem

Let $(\mathcal{C}, \Sigma, \nabla)$ be a pre-right triangulated category. Then the following are equivalent:

(1) ∇ *satisfies* (RTR4);

(2) ∇ *satisfies* (RTR4-1);

(3) ∇ *satisfies* (RTR4-2).

Our main results

We introduce the notion of homotopy cartesian square.

Definition

Let $(\mathcal{C}, \Sigma, \nabla)$ be a right triangulated category. Then a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle a} \\ C & \xrightarrow[b]{} & D \end{array}$$

is called homotopy cartesian if there exists a right triangle

$$A \xrightarrow{\binom{f}{g}} B \oplus C \xrightarrow{(a,\ -b)} D \xrightarrow{\delta} \Sigma A.$$

The morphism δ is called a differential of the homotopy cartesian square.

Our main results

Corollary

Let $(\mathcal{C}, \Sigma, \nabla)$ *be a right triangulated category. Then* (RTR4-3) *holds.*

(RTR4-3) *Every pair of morphisms* $X \xrightarrow{f} Y$ *and* $X \xrightarrow{g} X'$ *can be completed to a morphism*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle g}\downarrow & & \downarrow & & \| & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & \Sigma X' \end{array}$$

between right triangles such that the left hand square is homotopy cartesian and the composite $Y' \to Z \to \Sigma X$ *is a differential.*

Part IV: An application

Definition

Let $(\mathcal{C}, \Sigma, \nabla)$ be a right triangulated category. A full subcategory $\mathcal{S}$ of $\mathcal{C}$ is called a right triangulated subcategory if it satisfies the following conditions.

(1) $\mathcal{S}$ is extension closed. Namely, for any right triangle

$$X \to Y \to Z \to \Sigma X,$$

if $X, Z \in \mathcal{S}$, then $Y \in \mathcal{S}$.

(2) $\mathcal{S}$ is closed under Σ. Namely, $\Sigma\mathcal{S} \subseteq \mathcal{S}$.

Gabriel-Zisman localization

Let $(\mathcal{C}, \Sigma, \nabla)$ be a right triangulated category and $\mathcal{S}$ be a right triangulated subcategory.

$$\mathcal{R} := \{f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \mid \text{there exists a right triangle } X \xrightarrow{f} Y \to Z \to \Sigma X \text{ with } Z \in \mathcal{S}\}.$$

Denote by $\mathcal{C}[\mathcal{S}^{-1}]$ the Gabriel-Zisman localization of $\mathcal{C}$ with respect to $\mathcal{R}$.

Theorem

Let $(\mathcal{C}, \Sigma, \nabla)$ be a right triangulated category and $\mathcal{S}$ be a right triangulated subcategory. Then the category $\mathcal{C}[\mathcal{S}^{-1}]$ is a right triangulated category.

Thank you!