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Extra Polyloop-II and Its Representations

Oyeyemi Oluwaseyi Oyebola
oyebolao@brandonu.ca; oooyeyemi@gmail.com

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Joint work with: Tèmítọ́pẹ́ Gbọ́láhàn Jaíyéọlá
Kehinde Gabriel Ilori

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Introduction

In this study, we introduce a notable class of non-associative hyperalgebraic structure christened extra polyloop-II, and examine its algebraic properties. Extra polyloop-II is a special class of newly introduced non-associative hyperalgebraic structure, designated "Polyloop."

This work is focused on studying and investigating the hyperalgebraic properties and autotopic representation of extra polyloop-II. We also explore the notion of pseudo-automorphism in this hyperalgebraic structure.

In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

extra loop [F. Fenyves [8, 9]]

A loop $(G, \cdot)$ is said to be an extra loop if and only if it satisfies the identity $(xy \cdot z)x = x(y \cdot zx)$ for all $x, y, z \in G$. Identities (i) - (iii) were shown to be equivalent:

(i) $(xy \cdot z)x = x(y \cdot zx)$

(ii) $yz \cdot yx = y(zy \cdot x)$

(iii) $(x \cdot yz)y = xy \cdot zy)$ $\forall$x,y,z $\in G$

Consequently, in the study of non-associative algebraic hyperstructures, we present analogous results which characterize the extra polyloop-II identity with seven other equivalent identities.

Hypergroupoid, Hypergroup

In 1934, during the 8th Congress of Scandinavian Mathematicians, F. Marty [10] characterized hypergroups as a natural generalization of the idea of a group. The study of hyperstructure was further exhibited by P. Corsini [4]. In [12], polyquasigroups and polyloops were introduced, and extensively studied and some of their algebraic properties were established.

Definition

Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation (multivalued operation). The couple $(H, \circ)$ is known as a hypergroupoid.

Let G be a group and H be any subgroup of G. Then, $G/H = \{xH \mid x \in G\}$ becomes a hypergroup where the hyperoperation is defined in a usual manner:

$$aH \circ bH = \{cH \mid c \in aH \cdot bH\} \text{ for all } a, b \in G.$$

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Definition

A hypergroupoid $(H, \circ)$ is the pair of a non-empty set H with a hyperoperation $\circ : H \times H \to P(H) \backslash \{\emptyset\}$ defined on it. An hypergroupoid $(H, \circ)$ is called a semihypergroup if

(i) it obeys the associativity law $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v$$

An hypergroupoid $(H, \circ)$ is called a quasihypergroup if

(ii) it obeys the reproduction axiom $x \circ H = H = H \circ x$ for all $x \in H$.

A hypergroupoid $(H, \circ)$ is called a hypergroup if it is a semihypergroup and a quasihypergroup.

Definition

A hypergroupoid $(H,\circ)$ is called an H_v-group if it is a quasihypergroup and it obeys the weak associativity (WASS) condition

(iii) $x\circ(y\circ z)\cap(x\circ y)\circ z\neq\emptyset$ for all $x,y,z\in H$.

A hypergroupoid $(H,\circ)$ is called an Marty-Moufang hypergroup (H_m-group) if it is a quasihypergroup and it obeys the Moufang identity

(iv) $(x\circ y)\circ(z\circ x)=x\circ((y\circ z)\circ x)$ for all $x,y,z\in H$.

We often see the reproduction axiom used in the form: Given a, $b\in H$, there exist $x,y\in H$ such that $b\in a\circ x$ and $b\in y\circ a$. Hence, an hypergroup (of Marty) is equivalent to a multigroup of Dresher and Ore [7]

Definition

A hypergroup is a couple $(H, \circ)$, where $\circ : H \times H \longrightarrow P^*(H)$, such that the following conditions hold for all x, y, z of H:

1. $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$ which means that
$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v$$
2. $H \circ x = x \circ H = H$, where
$$H \circ x = \bigcup_{h \in H} h \circ x \text{ and } x \circ H = \bigcup_{h \in H} x \circ h$$
This condition is called *the reproduction axiom*.

Definition [Davvaz [6]]

A polygroup is a system $\wp =< P, \cdot, e, ^{-1} >$, where $e \in P$, $^{-1}$ is a unitary operation on P, $\cdot$ maps $P \times P$ into the non-empty subsets of P, and the following axioms hold for all $x, y, z \in P$:

(P1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(P2) $e \cdot x = x \cdot e = x$

(P3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

A polygroup is a special type of hypergroup.

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Definitions

Let $\mathscr{P} = (G, \cdot)$ be a polygroupoid such that $/ : G \times G \to P^*(G)$ and $\backslash : G \times G \to P^*(G)$.

(a) If (i) $y \in x \cdot (x \backslash y)$ (ii) $y \in x \backslash (x \cdot y)$ (iii) $y \in (y/x) \cdot x$ (iv) $y \in (y \cdot x)/x)$ then $(G, \cdot, \backslash, /)$ will be called a polyquasigroup.

(b) If $x \cdot e = e \cdot x = x$ for all $x \in G$ and $(G, \cdot, \backslash, /)$ is a polyquasigroup, then $(G, \cdot, \backslash, /, e)$ will be called a polyloop.

(c) $x \in x \cdot e = e \cdot x$ for all $x \in P$ and $(P, \cdot, \diagdown, \diagup)$ is a polyquasigroup. Then $(P, \cdot, \diagdown, \diagup, e)$ will be called a multiloop.

(d) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in P$ and $(P, \cdot, \diagdown, \diagup)$ is a polyloop. Then $(P, \cdot, \diagdown, \diagup)$ will be called an associative polyloop.

Definition

Let $(G,\cdot,/,\backslash,e)$ be a polyloop, then $(G,\cdot,/,\backslash,e)$ is called an extra polyloop-II if it satisfies the identity

$$(yx\cdot zx = y(xz\cdot x)$$

for all $x,y,z\in G$

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Theorem 1

Let $(G,\cdot,/,\backslash,e)$ be a polyloop then, $(G,\cdot,/,\backslash,e)$ is an extra polyloop-II iff any of the following is true in $(G,\cdot,/,\backslash,e)$ for all $x,y,z\in G$; $X,Y,Z\subseteq G$:

(i) $yX\cdot zX=(y\cdot Xz)X$.

(ii) $Yx\cdot zx=(Y\cdot xz)x$.

(iii) $yx\cdot Zx=(y\cdot xZ)x$.

(iv) $YX\cdot zX=(Y\cdot Xz)X$.

(v) $yX\cdot ZX=(y\cdot XZ)X$.

(vi) $Yx\cdot Zx=(Y\cdot xZ)x$.

(vii) $YX\cdot ZX=(Y\cdot XZ)X$.

Proof

Suppose $(G, \cdot, /, \backslash, e)$ is an extra polyloop-II then

$$yx \cdot zx = (y \cdot xz)x \tag{1}$$

Now, we show that (1) holds iff any of (i) - (vi) holds
L.H.S of (1):

$$yx \cdot zx = \bigcup_{a \in yx} a \cdot (zx) = \bigcup_{a \in yx, b \in zx} (a \cdot b).$$

R.H.S of (1):

$$(y \cdot xz)x = \bigcup_{p \in xz} (yp) \cdot x = \bigcup_{q \in \bigcup\limits_{p \in xz} yp} (q \cdot x).$$

$$\therefore \bigcup_{a \in yx, b \in zx} (a \cdot b) = \bigcup_{q \in \bigcup\limits_{p \in xz} yp} (q \cdot x) \tag{2}$$

Flexibility Laws for extra polyloop-II

Corollary

If $(G,\cdot,/,\backslash,e)$ is an extra polyloop-II, then the following are true in $(G,\cdot,/,\backslash,e)$ for all $x,z\in G$; $X,Z\subseteq G$

(i) $x\cdot zx = xz\cdot x$;

(ii) $X\cdot zX = Xz\cdot X$;

(iii) $x\cdot Zx = xZ\cdot x$;

(iv) $X\cdot ZX = XZ\cdot X$.

Remark

The identity (i) in the flexibility law of the Corollary above is true if $(G,\cdot,/,\backslash,e)$ is an extra loop. On the other hand, the identities (i) - (iv) in the Corollary above will jointly be called flexibility laws for extra polyloop-II.

Theorem 2

Let $(G,\cdot,/,\backslash,e)$ be an extra polyloop-II, then the following are true for all $x,y,z\in G$:

(i) $(G,\cdot,/,\backslash,e)$ is an inverse property polyloop i.e. $a\in ax\cdot x^{\rho}$ and $b\in x^{\lambda}\cdot xb\ \forall\ a,b,x\in G$.

(ii) $a\in aX\cdot X^{\rho}$ and $b\in X^{\lambda}\cdot Xb\ \forall\ a,b\in G;X\subseteq G$.

(iii) $Z\subseteq Zx\cdot x^{\rho}$ and $Y\subseteq x^{\lambda}\cdot xY\ \forall\ x\in G;Y,Z\subseteq G$.

(iv) $Q\subseteq QX\cdot X^{\rho}$ and $P\subseteq X^{\lambda}\cdot XP\ \forall\ P,Q,X\subseteq G$.

Proof

(i) Suppose $(G,\cdot,/,\backslash,e)$ is an extra polyloop-II, then we have:

$$yx\cdot zx=(y\cdot xz)x \tag{3}$$

Let $e\in yx \implies y\in e/x=\{x^{\lambda}\} \implies y=x^{\lambda}$. Then (3) becomes:

Proof Contd.

$$x^{\lambda}x \cdot zx = (x^{\lambda} \cdot xz)x$$

$$\implies zx \subseteq (x^{\lambda} \cdot xz)x$$

$$\implies z \in x^{\lambda} \cdot xz.$$

Next, let $e \in y \cdot xz \implies y \in e/xz = (xz)^{\lambda}$

$$\implies y \in (xz)^{\lambda} = (\overline{xz})$$

$$\implies y \in (\overline{xz})$$

Putting this in (3), we have:

$$((\overline{xz}) \cdot x)zx = ((\overline{xz}) \cdot xz)x$$

$$\implies x \in ((\overline{xz}) \cdot x)(zx)$$

$$\implies z\backslash zx \subseteq ((\overline{xz}) \cdot x)(zx)$$

Proof Contd.

$$\implies zx\bar{L}_z \subseteq zxL_{((\overline{xz})\cdot x)}$$

$$\implies \bar{L}_z \subseteq L_{((\overline{xz})\cdot x)}$$

$$\implies e\bar{L}_z \subseteq eL_{((\overline{xz})\cdot x)}$$

$$\implies z\backslash e \subseteq ((\overline{xz})\cdot x)$$

$$\implies z^\rho \in (\overline{xz})\cdot x$$

$$\implies x \in (\overline{xz})\backslash z^\rho$$

$$\implies x \in z^\rho \bar{L}_{(\overline{xz})} \subseteq z^\rho L_{(\overline{xz})^\lambda} (\because \bar{L}_a = L_{a^\lambda})$$

$$\implies x \in z^\rho L_{((\overline{xz}))^\lambda} = z^\rho L_{((xz)^\rho)^\lambda} = z^\rho L_{xz}$$

$$\implies x \in z^\rho L_{xz} \implies x \in xz \cdot z^\rho.$$

Hence, $(G, \cdot, /, \backslash, e)$ is an inverse property polyloop.

Autotopism of extra Polyloop-II

Theorem 3

Let $(G, \cdot, \backslash, /, e)$ be a polyloop, then $(G, \cdot, \backslash, /, e)$ is an extra polyloop-II if and only if $(R_X, \bar{L}_X R_X, R_X) \in AUT(G, \cdot, \backslash, /, e)$ for all $X \subseteq G$.

Proof

Let $(G, \cdot, \backslash, /, e)$ be a polyloop, then by Theorem 1 $(G, \cdot, \backslash, /, e)$ is an extra polyloop-II iff

$$yX \cdot zX = (y \cdot Xz)X$$

$$\implies yR_X \cdot zR_X = (y \cdot zL_X)R_X$$

Let $A_1 = yR_X \cdot zR_X$ and $A_2 = (y \cdot zL_X)R_X$
Suppose $q \in Xz \implies q \in \bigcup_{x \in X} xz$

$$\implies z \in \bigcup_{x \in X} x \backslash q \implies z \in X \backslash q \implies z \in q\bar{L}_X.$$

Conclusion and Further Studies

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It is worth noting that the generalization of classical quasigroup and loop is achieved by axiomatizing Polyquasigroup and polyloop. It was evident that polyquasigroup and polyloop are birthed from a non-commutative quasigroup and loop respectively.
A general sense of constructing an extra polyloop, characterized by seven equivalent identities, was investigated.
In future work, the extra polyloop will be investigated for other analogous algebraic properties that were satisfied by the extra loop in the classical sense. This will further be investigated and illustrated with examples.

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